Spectral rigidity of the round Earth

Inverse Days

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Slides and papers will appear at http://users.jyu.fi/~jojapeil

December 15, 2016

Prelude

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- There is a weaker version of the spectral problem: the spectral rigidity problem.
- Can we solve the simpler problem if we assume the Earth to be spherically symmetric?
- Yes!

Outline

The spectrum of free oscillations

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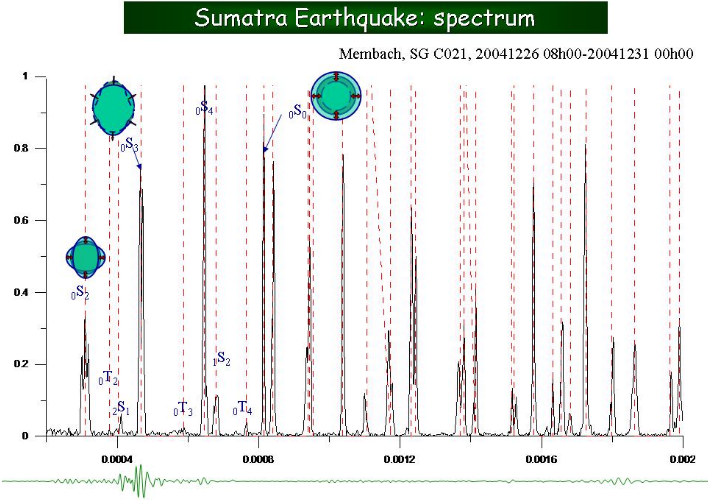
- The set of these frequencies is the spectrum of free oscillations.
- About 10 000 first frequencies are known.

The spectrum of free oscillations



Spectrum of free oscillations from an earthquake.

The spectrum of periodic orbits

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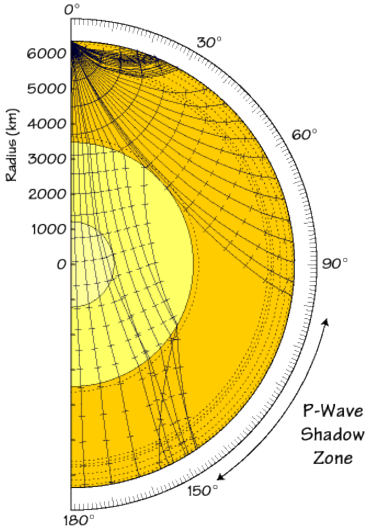
- Some of the wave paths are periodic. Every periodic wave path has a length (in time).
- The set of all lengths of periodic seismic wave paths is the "length spectrum" of the Earth.
- Originally the length spectrum was just a mathematical tool, but it turns out it can be measured directly. Campillo and Poli (unpublished) have found a way to directly measure the length spectrum from the surface using deep earthquakes.

The spectrum of periodic orbits



Seismic wave paths and the P-wave shadow zone. (Wikimedia Commons)

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We want to reconstruct the Earth in the natural Cartesian coordinates.

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- We model the Earth as a Riemannian manifold with boundary.
- In practice, the Earth is the closed unit ball $M = \bar{B}(0,1) \subset \mathbb{R}^3$. The anisotropic wave speed is modeled with a Riemannian metric g on M.
- Physically, this corresponds to omitting S-waves and including only elliptic anisotropy.

The spectrum of the Laplacian

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- If the wave speed is isotropic, then $g = c^{-2}e$ and the Laplace–Beltrami operator in dimension n is

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$$\Delta_g u(x) = c(x)^{4-n} \operatorname{div}(c(x)^{n-2} \nabla u(x)).$$

- The spectrum of free oscillations is the Neumann spectrum of the Laplace–Beltrami operator Δ_g.

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- The length spectrum of (M, g) is the set of all lengths of the periodic broken rays.

Outline

1. Seismic spectral data
2. Spectra of a manifold with boundary
3. Different forms of uniqueness
   - Difficulties
   - Diffeomorphisms and coordinates
   - Global uniqueness
   - Local uniqueness
   - Spectral rigidity
4. Spherical symmetry
5. The main results
6. Bonus content: Ray transforms in low regularity and drums

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- Proving this conjecture is difficult for two reasons:
    1. The required tools do not yet exist on general manifolds with boundary.
    2. The conjecture is false.

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- The main obstacle to uniqueness is that there are no preferred coordinates.
- If $\phi \colon M \to M$ is a diffeomorphism, then (M, g) and $(M, \phi^* g)$ give the same spectrum.
- We can take any change of coordinates whatsoever and use it to distort the metric, but the spectrum stays the same.
- Physically: There are preferred and natural Cartesian coordinates. But the anisotropic model is not "sensitive to the underlying Euclidean geometry", so the Cartesian coordinates cannot be recognized. It is impossible to find the metric (anisotropic sound speed) in Cartesian coordinates from spectral data.

Global uniqueness

Problem

Let g_1 *and* g_2 *be two Riemannian metrics on a manifold* M *with boundary. If they give the same spectrum, is there a diffeomorphism* $\phi\colon M \to M$ *so that* $g_1 = \phi^* g_2$*?*

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This is too hard.

Local uniqueness

Problem

Let g_1 and g_2 be two Riemannian metrics on a manifold M with boundary. Suppose g_1 is very close to g_2. If they give the same spectrum, is there a diffeomorphism $\phi\colon M \to M$ so that $g_1 = \phi^* g_2$?

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Spectral rigidity

Problem

Let g_s be family of Riemannian metrics on a manifold M with boundary, depending on a parameter $s \in (-\varepsilon, \varepsilon)$. If they all give the same spectrum, are there a diffeomorphisms $\phi_s \colon M \to M$ so that $g_0 = \phi_s^* g_s$?

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This is within reach!

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  - Negatively curved surfaces: Guillemin–Kazhdan 1980.
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- We have adapted similar ideas of proof to manifolds with boundary.

Outline

Spherically symmetric manifolds

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- If g is a rotation invariant Riemannian metric on M, there is a radial (more complicated if $n = 2$) diffeomorphism $\phi \colon M \to M$ so that $\phi^* g$ is radially conformally Euclidean.
- The Earth is spherically symmetric to a good approximation, but the best radial model might not be conformally Euclidean. After a radial change of coordinates the metric becomes conformal — and Cartesian coordinates are lost.

The Herglotz condition

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Definition

A radial sound speed $c(r)$ satisfies the Herglotz condition if

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- The manifold is non-trapping and has strictly convex boundary.

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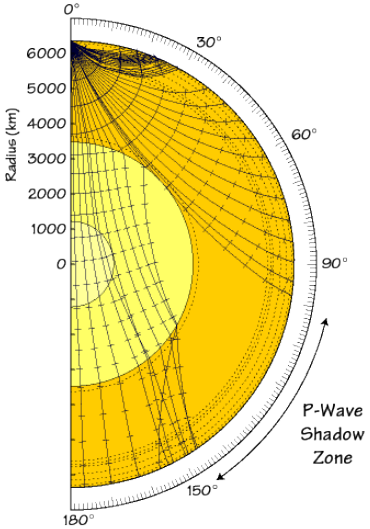
- In addition, the shear wave speed vanishes in the liquid outer core.
- Apart from these problems (jumps and liquid) both shear and pressure wave speeds do satisfy the Herglotz condition everywhere.

The Herglotz condition



Some P-waves are trapped in the outer core. (Wikimedia Commons)

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   - Length spectral rigidity
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Let M be the closed unit ball in $\mathbb{R}^3$. Let $c_s(r)$ be a family of radial wave speeds depending C^∞-smoothly on both $s \in (-\varepsilon, \varepsilon)$ and $r \in [0, 1]$. Assume each c_s satisfies the Herglotz condition and a generic geometrical condition.

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This simple model of the round Earth is spectrally rigid!

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Let M be the closed unit ball in $\mathbb{R}^3$. Let $g_s(r)$ be a family of rotation invariant metrics depending C^∞-smoothly on both $s \in (-\varepsilon, \varepsilon)$ and $r \in [0, 1]$. Suppose each g_s is non-trapping with strictly convex boundary and assume a generic geometrical condition.

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If the spectra of the Laplace–Beltrami operators Δ_{g_s} are all equal, then there is a family of radial diffeomorphisms $\phi_s \colon M \to M$ so that $\phi_s^* g_s = g_0$ for all s. That is, the manifolds (M, g_s) are isometric.

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Let M be the closed unit ball in $\mathbb{R}^n$, $n \geq 2$. Let $c_s(r)$ be a family of radial wave speeds depending $C^{1,1}$-smoothly on both $s \in (-\varepsilon, \varepsilon)$ and $r \in [0, 1]$. Assume each c_s satisfies the Herglotz condition and a generic geometrical condition.

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If the length spectra of the manifolds (M, g_s) are all equal, then there is a family of radial (or more general if $n = 2$) diffeomorphisms $\phi_s \colon M \to M$ so that $\phi_s^* g_s = g_0$ for all s. That is, the manifolds (M, g_s) are isometric.

Ideas behind the proof

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Lemma (Trace formula)

Let $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ *be the positive eigenvalues of the Laplace–Beltrami operator. Define a function* $f : \mathbb{R} \to \mathbb{R}$ *by*

$$f(t) = \sum_{k=0}^{\infty} \cos\left(\sqrt{\lambda_k} \cdot t\right).$$

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The function $f(t)$ *is singular precisely at the length spectrum.*

In particular, the spectrum determines the length spectrum.

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Similar “trace formulas” and related results are known on closed manifolds (eg. Duistermaat–Guillemin 1975) and a weaker version on some manifolds with boundary (eg. Guillemin–Melrose 1979).

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Corollary

Spectral rigidity follows from length spectral rigidity.

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In particular, if the length spectrum does not depend on s*, then* $\frac{\mathrm{d}}{\mathrm{d}s} c_s^{-2}$ *integrates to zero over all periodic broken rays.*

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This conclude the proof.

Remark: No proof works without spherical symmetry.

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X-ray transforms

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Earlier similar results:

- The X-ray transform (Radon et al.): Euclidean metric (c is constant).
- Mukhometov, 1977: Smooth simple metrics (simplicity is stronger than Herglotz).
- Sharafutdinov, 1997: C^∞ metrics and C^∞ functions.

Periodic broken ray transforms

Periodic broken ray transforms

Theorem (de Hoop–I.)

Let M *be a rotation symmetric non-trapping manifold with a* $C^{1,1}$ *metric and strictly convex boundary and dimension at least three. Assume that there are not too many conjugate points at the boundary. The integrals of a function* $f \in L^p(M)$, $p > 3$, *over all periodic broken rays determines the even part of the function.*

Very little can be recovered of the odd part.

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Tools used:

- Planar average ray transform.
- Abel transform.
- Funk transform.
- Fourier series.

Hearing the shape of a drum

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- If we know that the metric on $M \subset \mathbb{R}^n$ is (conformally) Euclidean, this ambiguity due to diffeomorphisms goes away.
- This is why the answer to Kac's famous question "Can you hear the shape of a drum?" is not trivially "No!".
- ... but it is non-trivially "No!" if there are no geometrical restrictions.

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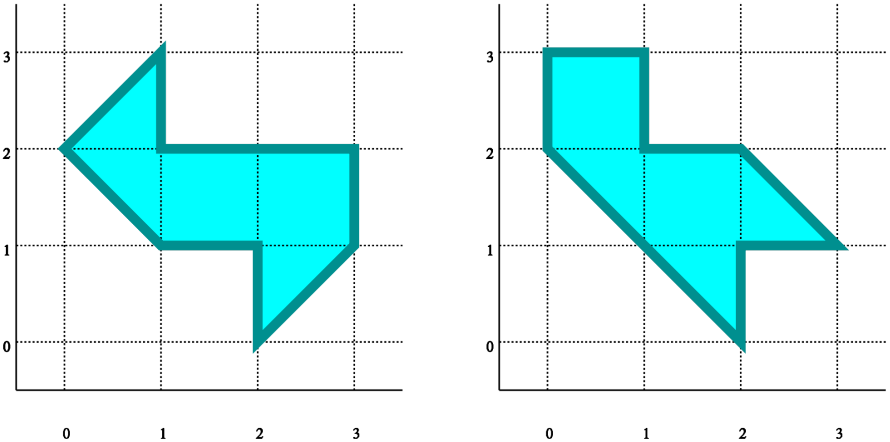

These two drums sound exactly alike. (Wikimedia Commons)

End

Slides and papers will appear at http://users.jyu.fi/~jojapeil.

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Thank you.