Quantum mechanical tomography and neutrino oscillation

Inverse Days 2015

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Outline

1. Neutrino oscillation
   - Neutrinos
   - Schrödinger's equation and mass
   - Oscillation
   - Matter effects
2. Quantum mechanical tomography

Neutrinos

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  - the tau neutrino ν_τ
- Neutrino masses are $\lesssim 1$ eV. (Natural units: $c = \hbar = 1$.) For comparison, the masses of an electron and a neutron are $5 \cdot 10^5$ eV and $2 \cdot 10^9$ eV.

Schrödinger's equation and mass

- We can think of neutrinos as point particles. The state of a neutrino is

$$\Psi(x,t) = \begin{pmatrix} \Psi_{\nu_e}(x,t) \\ \Psi_{\nu_\mu}(x,t) \\ \Psi_{\nu_\tau}(x,t) \end{pmatrix} \in \mathbb{C}^3.$$

All flavours are collected into a single state. If the state is measured, the probability of finding ν_e at (x,t) is $|\Psi_{\nu_e}(x,t)|^2$.

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- We need an equation of motion for $\Psi(x,t)$.

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- In vacuum we obtain the ultrarelativistic Schrödinger equation

$$i\partial_t \Psi(x,t) = (p + \frac{1}{2p} M^2)\Psi(x,t),$$

where M is the neutrino mass matrix.

Schrödinger's equation and mass

- If the neutrinos ν_e, ν_μ, ν_τ had masses $m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau}$, the mass matrix would be

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The basis can be changed via the PMNS matrix U.

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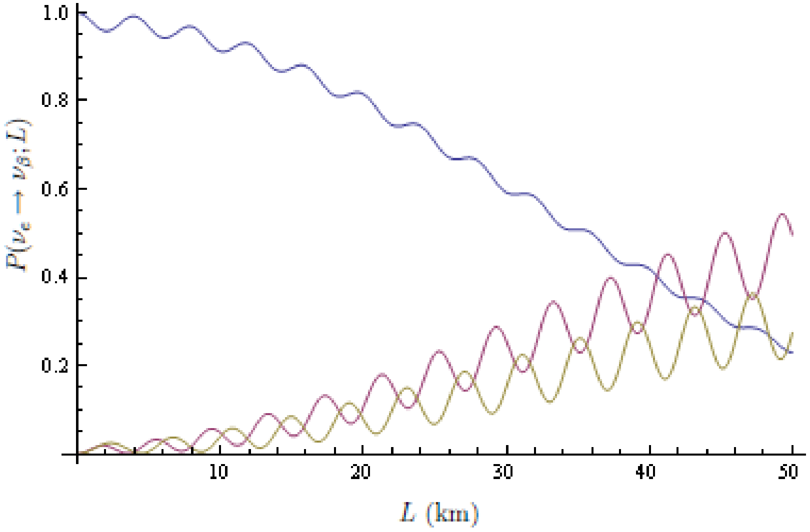
- Different mass states have slightly different velocities and therefore they slowly drift apart. They lose coherence and no longer oscillate. We will focus on coherent phenomena.

Oscillation



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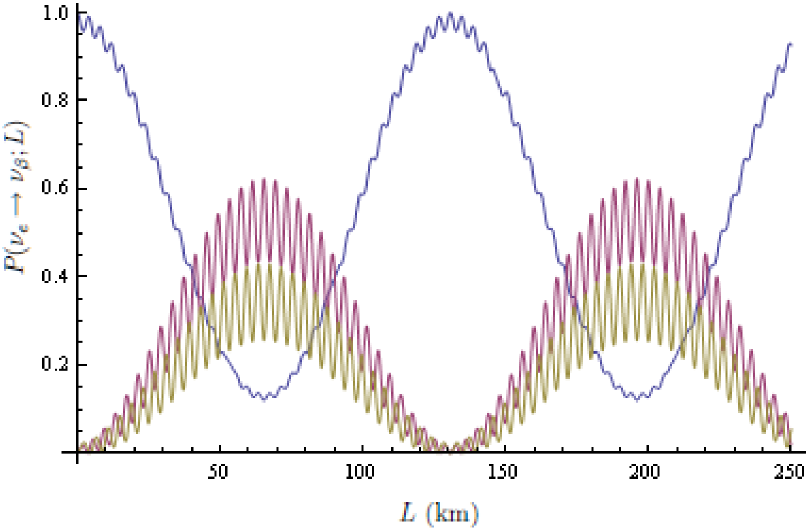

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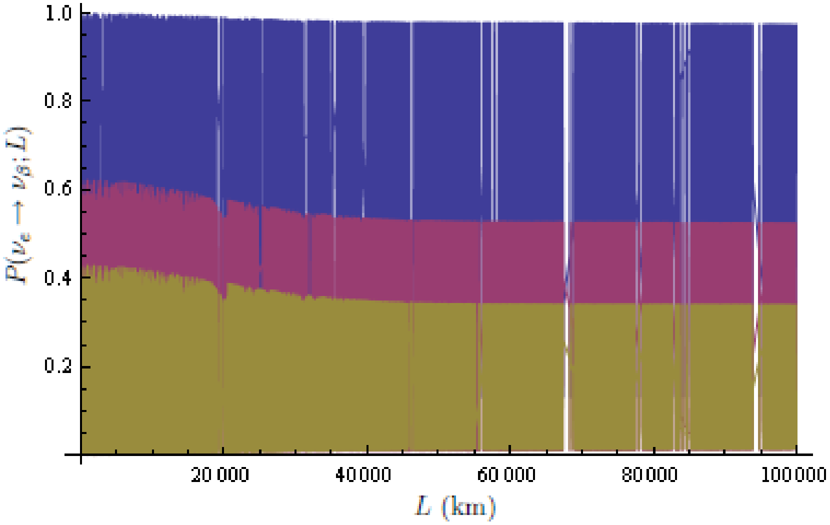

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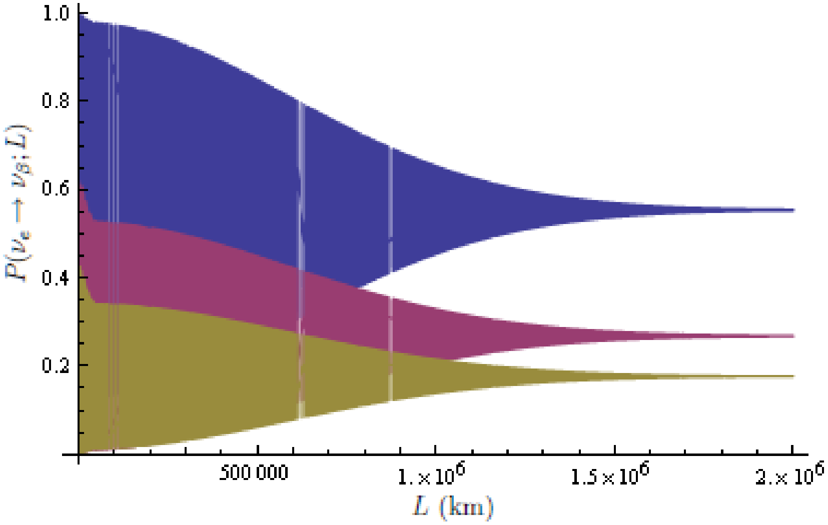

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- Interactions with the background change the oscillation.
- **Question:** Can we reconstruct the background from neutrino oscillation data?

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- The Hamilton matrix $H(x) \in \mathbb{C}^{N \times N}$ depends on $x \in \Omega$.
- The time evolution of a state $\Psi(t) \in \mathbb{C}^N$ along the trajectory $t \mapsto x_0 + tv$ is given by $i\partial_t \Psi(t) = H(x_0 + tv)\Psi(t)$.

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- For neutrinos, both sets are the flavor basis (three orthogonal unit vectors in $\mathbb{C}^3$).
- We measure this number $|\langle f, \Psi(T)\rangle|^2$ for all initial states $\Psi(0)$, all reference states f and all trajectories through Ω.

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- For positive results we need to assume “ideal data”.
  - Example: All states in $\mathbb{C}^N$ are available as initial and reference states.
  - Non-example: Neutrino oscillation data.

The result

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Theorem (I., 2015)

Let $\Omega \subset \mathbb{R}^n$*,* $n \geq 3$*, be a bounded convex domain. Let* $N \geq 1$ *be an integer and suppose* $H, \tilde{H} \in C^{1,\alpha}(\bar{\Omega}, \mathbb{C}^{N \times N})$ *for some* $\alpha > 0$ *are pointwise hermitean.*

Assume ideal data in the described measurements. The two Hamiltonians H *and* $\tilde{H}$ *give the same data if and only if* $\tilde{H} = H + \phi I$ *for a scalar function* ϕ*.*

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- The weighted integral of a function $f \colon \bar{\Omega} \to \mathbb{C}^M$ over a smooth unit speed curve $\gamma \colon [0, T] \to \bar{\Omega}$ is

$$\int_0^T W(\gamma(t), \dot{\gamma}(t)) f(\gamma(t)) \mathrm{d}t.$$

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Theorem (I., 2015)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded convex domain. If W is $C^{1,\alpha}$ for some $\alpha > 0$ and pointwise invertible, then a continuous $f \colon \bar{\Omega} \to \mathbb{C}^M$ is uniquely determined by its weighted integrals over all straight lines.

End

J. Ilmavirta: **Coherent quantum tomography**, preprint, arXiv:1507.00558.

Slides and papers will appear at `http://users.jyu.fi/~jojapeil`.

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Thank you.