JYVÄSKYLÄN YLIOPISTO
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Functions of constant X-ray transform

Inverse Days

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Based on joint work with

Gabriel Paternain

Outline

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  whenever this makes sense.
- Everything is set up in the domain; no lines or points outside Ω are considered.

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- On manifolds there is a Pestov–Uhlmann range characterization (2008).
- Our question is: **Is a non-zero constant function $\Gamma_\Omega \to \mathbb{R}$ in the range of I?**

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- Conclusion: Functions with piecewise constant X-ray transform can be transparent in some sense.

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- Geodesics get short near the boundary, so f has to blow up at the boundary.
- If If is constant, so is I^*If.

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Theorem (Monard–Nickl–Paternain, to appear)

The normal operator is a bijection

$$I^*I\colon d^{-1/2}C^\infty(M) \to C^\infty(M)$$

for a simple Riemannian manifold M, where d is a non-vanishing smooth function which coincides with distance to the boundary near the boundary.

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- $If =$ const. $\implies I^*If =$ const. $\implies f \in d^{-1/2}C^\infty(M)$.
- If there is f with $If \equiv 1$, it is unique.

Outline

1. The X-ray transform
2. Examples
   - Euclidean ball
   - Radially symmetric Riemannian manifold
   - Piecewise constant X-ray transform
3. Characterization of Euclidean domains
4. Improvements

Euclidean ball

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- In the Euclidean unit ball the function

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- There are no other such functions in the ball.
- The function is exactly the same in every dimension, and radially symmetric.

Radially symmetric Riemannian manifold

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- Assuming the Herglotz condition, there is a function a so that $f(x) = a(|x|)$ satisfies $If \equiv 1$, and a can be computed from c explicitly.
- If the Herglotz condition fails, there is no such function.

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- Any sum of functions of this form has a piecewise constant X-ray transform and is "transparent" in the sense mentioned earlier.
- This is possible in any domain, but the X-ray transform is not constant and the function itself has interior singularities.
- The same construction does not seem to be possible on all simple manifolds.

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1. *There is a function $f \in L^1(\Omega)$ for which $If : \Gamma_\Omega \to \mathbb{R}$ is a non-zero constant.*
2. *Ω is a ball.*

A sketchy proof: two dimensions

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- Therefore Ω has the same width in all directions.

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- Combining this new information with constant width shows that Ω must be a disc.

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The theorem for $n \geq 3$ follows from this lemma, as the only bounded domains with umbilical boundary are balls.

A sketchy proof: higher dimensions

- To prove the lemma, recall the boundary behavior of the function:

$$f(x) = d(x, \partial M)^{-1/2} w(x)$$

near ∂M, where $w \in C^\infty(\bar{M})$.

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- We get

$$\sqrt{2/\mathrm{I\!I}(v, v)}\pi w(x) = 1$$

for all v, so the second fundamental form is independent of direction. Thus the boundary is umbilical at x.

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1. The X-ray transform
2. Examples
3. Characterization of Euclidean domains
4. Improvements
   - Local result
   - Radon transforms
   - Simple manifolds

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- The same can be asked about the d-plane Radon transform in $\mathbb{R}^n$ for any $1 \leq d < n$.
- These questions only make sense in Euclidean geometry.
- Ramya Dutta and Suman Kumar Sahoo verified that in all these cases the domain can only be a ball.

Simple manifolds

Simple manifolds

Question

Let M be a simple Riemannian manifold with $\dim(M) \geq 2$. If there exists a function $f \in L^1(M)$ that integrates to 1 over every geodesic, is it true that M is spherically symmetric?

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