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Inverse problems with neutrinos

Inverse Problems in Analysis and Geometry

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Based on joint work with
Gunther Uhlmann

Topics

- The physics of neutrinos.
- Inverse problems with neutrinos.

Outline

Matter in the Standard Model

There are two sectors of matter particles (not force carriers) in SM:

Quarks

		electric charge	
		$q = -\frac{1}{3}e$	$q = +\frac{2}{3}e$
generation	1	d, down	u, up
	2	s, strange	c, charm
	3	b, bottom	t, top

Leptons

		electric charge	
		$q = -e$	$q = 0$
generation	1	e, electron	ν_1
	2	μ, muon	ν_2
	3	τ, tau	ν_3

Generations/flavors only differ by mass, and all 12 masses are different.
The charge jump is $1e$ in both sectors.

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The linear combination of neutrinos ν_1, ν_2, ν_3 that couples to the electron is called the electron neutrino, $\nu_e = \sum_{i=1}^{3} U_{ei}\nu_i$.

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This *neutrino oscillation* is a kinematic phenomenon.

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Kinematically we can thus treat neutrinos as perturbations of photons (which have $v = c$), so they are well modeled as *ultrarelativistic Jacobi fields along null geodesics*.

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Neutrinos (like photons) travel cosmological distances and they are not disturbed by our local electromagnetic fields.

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Neutrino oscillation

The state space of a neutrino is 3-dimensional, and the state (in the flavor basis) is

$$\psi(t) = \begin{pmatrix} \psi_e(t) \\ \psi_\mu(t) \\ \psi_\tau(t) \end{pmatrix} \in \mathbb{C}^3.$$

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Semiclassical description: A classical point particle that carries a quantum state.

Neutrino oscillation

In vacuum we have just the free Hamiltonian

$$H_0 = \frac{1}{2E} U_{\mathrm{PMNS}} \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} U^*_{\mathrm{PMNS}}.$$

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In a medium we have $H = H_0 + N_e A$, where

$$A = 2\sqrt{2} E G_F \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where N_e is the electron number density.

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Phase information is lost, so multiples of the identity matrixare invisible.

The inverse problem

Question

If we can prepare and measure the initial and final states of neutrinos passing through an object, can we find the electron number density $N_e(x)$?

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Theorem [I., 2016]

Suppose we do the measurements above for all straight lines through a nice domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ with a hermitean Hamiltonian field $H \colon \Omega \to \mathbb{C}^{n \times n}$.

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The conclusion is more than strong enough for the physical problem, but we also assume too big a library.

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Idea

Neutrino worldlines are only almost null — remember the ultrarelativistic Jacobi fields. They break this conformal symmetry infinitesimally.

The result

Theorem (Kurylev–Lassas–Uhlmann, 2018)

Measurements of light cones in an open subset of the spacetime determine the geometry and conformal class of the spacetime in the lightlike past of the measurement set.

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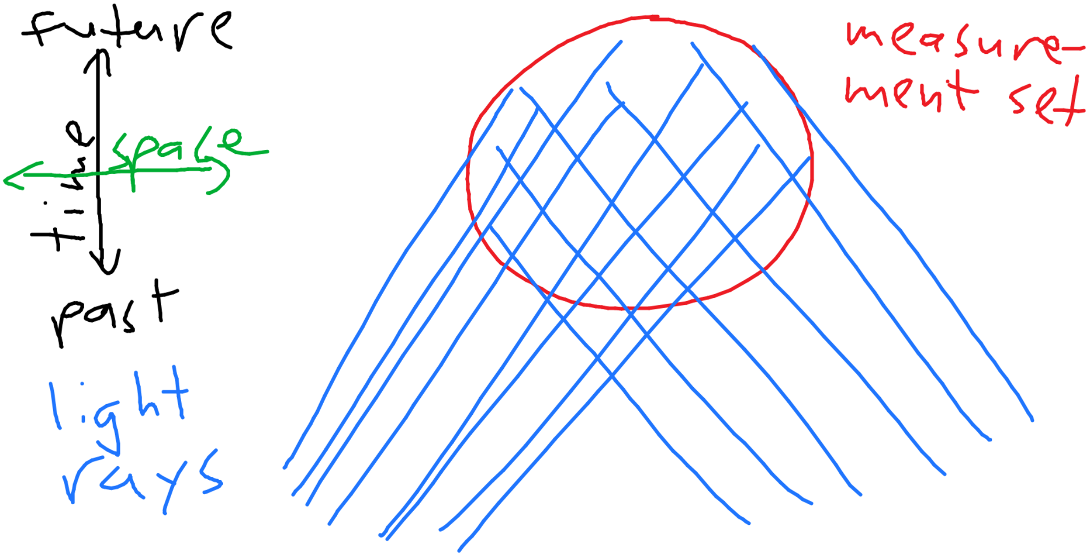
Theorem (I.–Uhlmann, 2021)

Suppose the conformal class is known. Measurements of (perturbative) neutrino cones in an open subset of the spacetime determine the conformal factor in the lightlike past of the measurement set.

Photons and neutrinos together determine the full geometry of the visible part of the spacetime!

Visible past



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- Neutrinos are a tiny bit slower than photons ($v \approx (1 - 10^{-20})c$) but are released a little earlier.
- The neutrino cone depends on the motion of the dying star, the light cone does not.

The two cones

Not the normal kind of normal

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  $\implies$ The inverse problem becomes simple!

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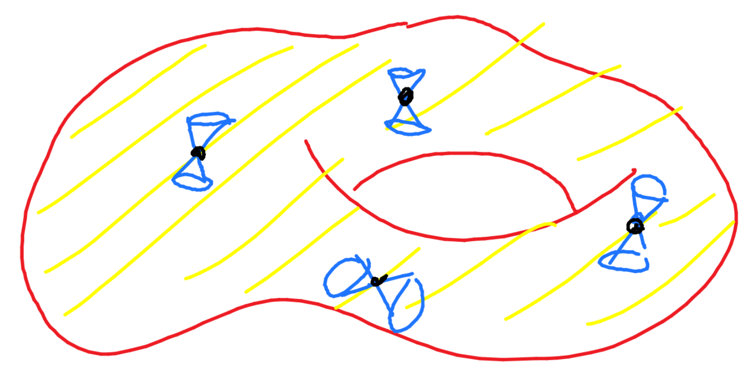
- A conformal change of the metric tensor leaves the light cones unchanged.
- Particles travelling at the speed of light only care about the conformal class. They have no sense of scale, local or global!
- Mass is the ability to sense scale.

Conformal gauge

light cone bundle $= \{v \in TM; v \cdot v = 0\}$

$\leftrightarrow$ possible photon directions

$\leftrightarrow$ conformal class



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Lessons

Neutrinos. . .

- . . . interact very weakly in all respects.
- . . . oscillate between the flavors ν_e, ν_μ, ν_τ.
- . . . have a tiny mass, but that is enough to sense scale and fix a conformal factor.
- . . . are well modeled by ultrarelativistic Jacobi fields along light rays.
- . . . can see what other particles cannot.
- . . . are fun.