JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

# Inverse problems with one sensor on Mars

Geo-Mathematical Imaging Group
2019 project review

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Based on joint work with
Maarten de Hoop and Vitaly Katsnelson

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- Assume perfect measurements from a single ideal seismometer. What can you say for sure and is there an inversion algorithm?
- Identifying useful data sets can help future mission planning.
- Grand goal: A mathematical theory of seismic planetary exploration.

# Outline

1. Seeing the radial Martian mantle with InSight
2. Seeing an entire planet

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- Mars is roughly spherically symmetric. There are reliable ways to reconstruct a radial model of the (upper) mantle from a single station. (The mantle determines the CMB.)
- I will ignore noise, model errors, finiteness, stability, and many other practical things.

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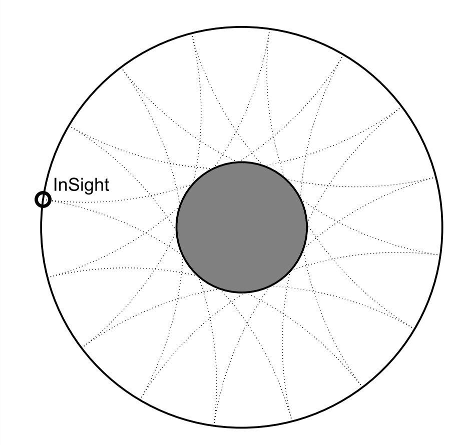
- Linearized data: Pairs of periodic broken rays and integrals over them. Uknown: Variations of wave speed (a function).

# Method A: Linearized travel time tomography




Periodic seismic ray reflecting on the surface and CMB.

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Solving the linearized problem gives an iterative algorithm to solve the nonlinear one.

(Uniqueness should be provable for the non-linear one, too.)

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- The spectrum of free oscillations can be measured from any single point.
- Mathematically, the spectrum of free oscillations corresponds to the Neumann spectrum of the Laplace–Beltrami operator on a manifold.

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**Question**

*If a family of wave speeds $c_s(r)$ have the same spectrum, are the equal? Is the (Martian) mantle spectrally rigid?*

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This is based on linearization. With a trace formula one ends up showing that the Martian mantle is length spectrally rigid.

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- Surface waves will come from the event to InSight two ways along the great circle containing the impact site and InSight.
- If there are no other events on the same great circle around the same time, we can measure the time difference $\delta$.
- Multiple arrivals or a priori information tells the time $T$ around the great circle.

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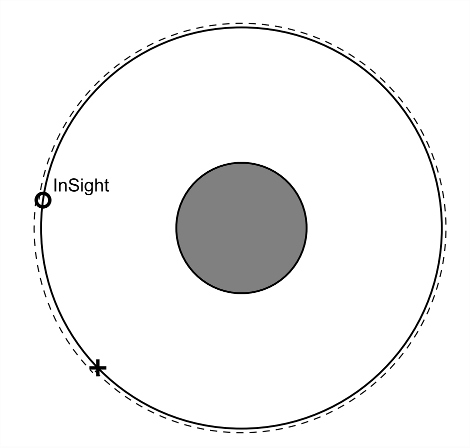


Two surface wave arrivals from the same event.

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- This was all done on surface, and it gives rise to interior data: Now using body waves we know the travel time between InSight and the source.
- To get here, we needed to assume spherical symmetry only on the surface, but the arising problem is easiest to solve if the symmetry extends inside.

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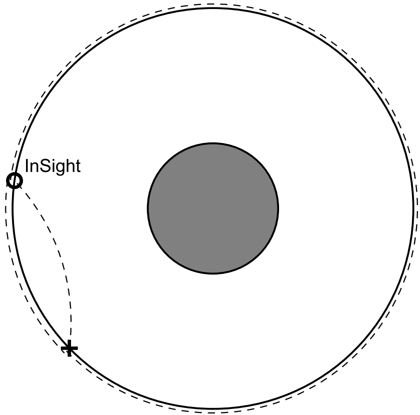


The body wave whose initial point and time were located with surface waves.

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- The linearized problem is X-ray tomography (or an Abel transform), and can also be solved explicitly. (e.g. de Hoop–I., 2017)

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- If the three reconstructions all work and give similar results, we can be quite confident.
- This gives us an isotropic radially symmetric reference model of the mantle, which is a stepping stone towards deeper and finer structure.

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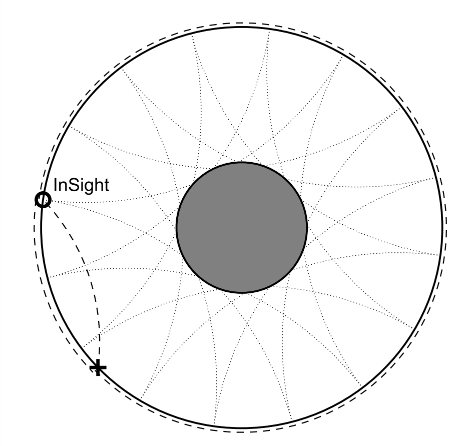


Three ways to see the mantle from InSight.

# Summary

- A: From noise correlations to (linearized) travel times.
- B: From spectrum to length spectrum.
- C: Meteorites; body wave data calibrated by surface waves.

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- It remains to be proven that this is indeed the only obstruction to uniqueness.

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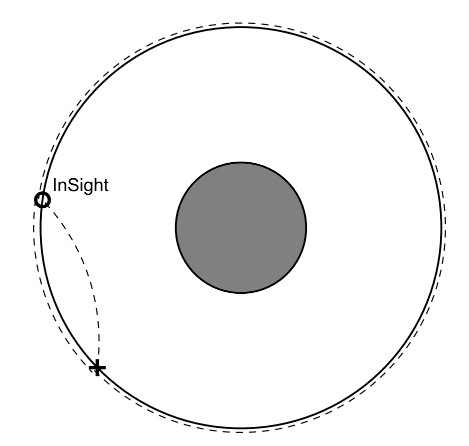
- This leads to travel time data: The travel times (geometrically: distances) are known from all points on the surface to a single fixed point.

# Half-local X-ray tomography

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This is possible at least in Euclidean geometry or with real analytic perturbations but always unstable.

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- What is the geometry of periodic broken rays?

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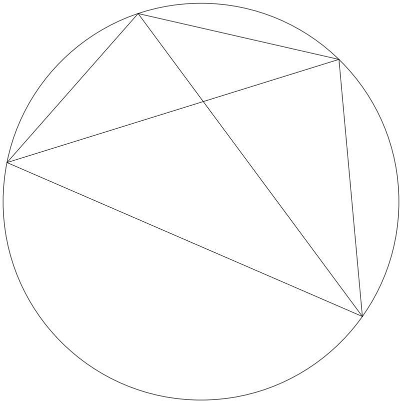

Boundary distance rigidity: Do the distances between all boundary points determine the meometry?

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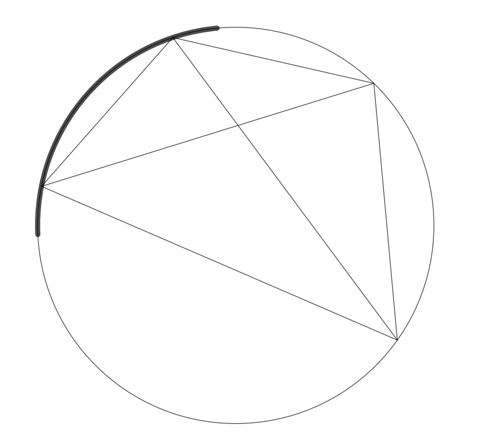

We have an accessible region — a measurement array.
The size is exaggerated.

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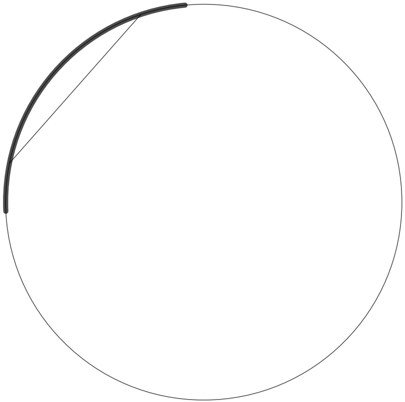
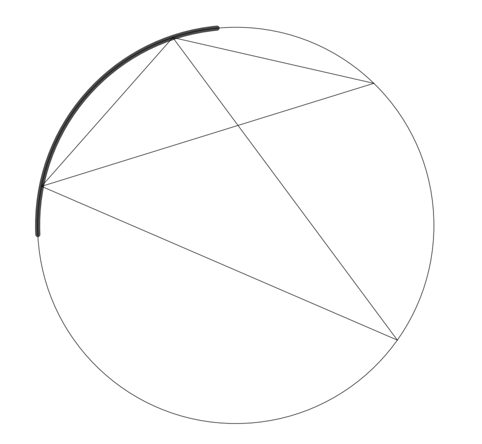


In the local boundary distance problem one knows the distances between the points in the small set and wants to find the geometry near that set.

# A theory?



The “half-local” boundary distance data has more information and one wants to reconstruct the whole geometry.

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