JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

Dix's inverse problem on elastic Finsler manifolds

Geo-Mathematical Imaging Group
2019 project review

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April 29, 2019

Based on joint work with
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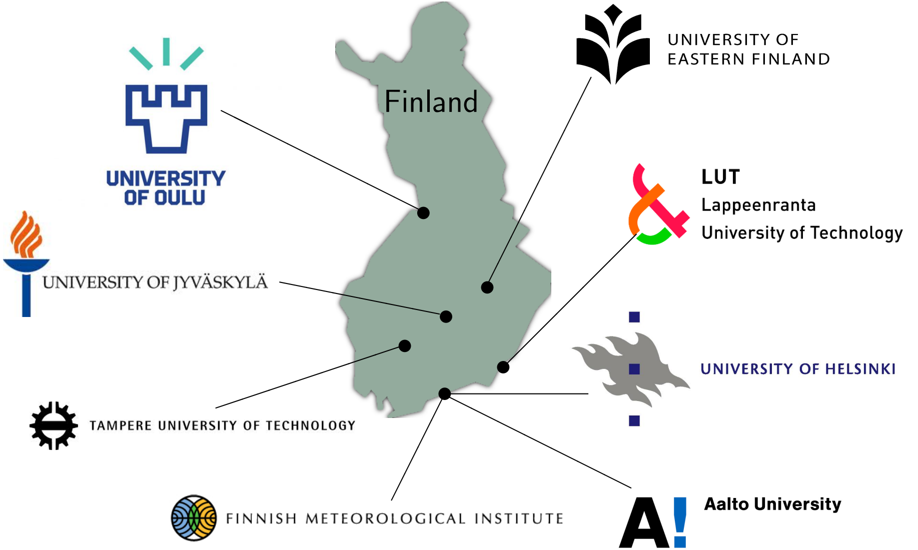
Finnish Centre of Excellence in Inverse Modelling and Imaging

2018-2025

CENTRES OF EXCELLENCE
IN RESEARCH



Goals

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- Understanding how and why to model elastic waves in terms of Finsler geometry.

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- Seeing how geometrical tools can be used to solve Dix's problem in great generality.

Outline

Elastic waves

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In linear elasticity the displacement vector $u(x,t)$ satisfies the elastic wave equation

$$\partial_j[c_{ijkl}(x)\partial_k u_l(x,t)] - \rho(x)\partial_t^2 u_i(x,t) = 0,$$

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The singularities can be described in geometrical terms as point particles or wave fronts.

Propagation of singularities

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The slowness vector p (inverse phase velocity), position x, and polarization A of the singularity have to satisfy

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The possible values of p are on the slowness surface defined by the equation

$$\det[\Gamma(x,p) - I] = 0.$$

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Microlocal analysis (or physical reasoning) tells how singularities propagate: They follow a Hamiltonian flow related to the slowness surface.

If the slowness surface is smooth and strictly convex (qP!), then this coincides with the geodesic flow of a Finsler manifold, where the Finsler metric is the Legendre transform of the slowness surface.

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If the slowness surfaces are ellipsoids, we have a Riemannian manifold. But we do not assume any kind of isotropy!

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- The slowness surface corresponds to a co-Finsler metric.
- Taking the Legendre transform gives a Finsler metric which corresponds to group velocities.
- The singularities of elastic waves (= elastic waves as point particles) follow geodesics of this Finsler geometry.

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Gains:

1. Freedom to have any shape of slowness surfaces, not necessarily arising from Hookean elasticity.
2. Access to powerful tools of differential geometry.

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Not all Finsler metrics arise from a stiffness tensor. When does $\phi^* F$ do?

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We make no assumptions on symmetry, isotropy, or homogeneity.

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Theorem (de Hoop–I.–Lassas)

Let (M, F) be any Finsler manifold and $U \subset M$ an open set. Suppose F is fiberwise analytic. If we know the smooth metric "spheres" (wave fronts from point sources) together with radii (travel times) in the set U, we can find the universal cover of (M, F).

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All Finsler metrics arising from elasticity are fiberwise analytic. This is a good example of useful additional structure that comes from good modelling — the model is general enough to handle all anisotropy but narrow enough to exclude nonphysical oddities.

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- We can find the geometry along any geodesic, and this information must be turned into a global geometrical description. This requires using analyticity and building an atlas (a collection of coordinate charts).

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