Broken ray tomography in the disk

Inverse Days 2012

Joonas Ilmavirta

University of Jyväskylä

18.12.2012

Broken rays

- In a (bounded smooth) Euclidean domain Ω a *broken ray* is a geodesic which starts and ends at $\partial\Omega$ and may have reflections at $\partial\Omega$.
- For a fixed set $E \subset \partial\Omega$, consider the following problem: If we know the integral of an unknown function $f : \Omega \to \mathbb{R}$ over every broken ray from E to E, what can we say about f?

Broken ray transform

- Given Ω and $E \subset \partial\Omega$, the set of broken rays from E to E is denoted by Γ_E.
- For $f \in C(\bar{\Omega}, \mathbb{R})$ and $\gamma \in \Gamma_E$ we define

$$\mathcal{G}f(\gamma) = \int_\gamma f \mathrm{d}s.$$

The map $\mathcal{G} : C(\bar{\Omega}, \mathbb{R}) \to B(\Gamma_E, \mathbb{R})$ is the *broken ray transform*.

- Full reconstruction is possible $\Leftrightarrow$ $\mathcal{G}$ is injective.

Motivation

- Some partial data problems for the Calderón problem can be reduced to injectivity of the broken ray transform (Salo&Kenig 2012).
- From partial Dircihlet to Neumann data for the magnetic Schrödinger equation one can recover the integrals of the electromagnetic potential over broken rays (Eskin 2004).
- The broken ray problem may also arise more directly in X-ray-like imaging with reflections.

Tomography from a singleton

Theorem

Suppose that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is the unit ball, $E \subset \partial\Omega$ is a singleton and $f \in C(\bar{\Omega}, \mathbb{R})$. Then we can recover $f(0)$ from the broken ray transform $\mathcal{G}f$.

Remark: When $n = 2$, we can recover the integral average of f over any circle $\partial B(0, r)$, $r \in (0, 1]$.

Tomography from an open set

Theorem

Suppose that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is the unit ball, $E \subset \partial\Omega$ is open and $f : \bar{\Omega} \to \mathbb{R}$ is uniformly quasianalytic in the angular variable. Then we can recover f everywhere from the broken ray transform $\mathcal{G}f$.

For example, functions of the form (in polar coordinates in the plane)

$$f(r,\theta) = a_0(r) + \sum_{k=1}^{K} [a_k(r)\cos(k\theta) + b_k(r)\sin(k\theta)]$$

are uniformly quasianalytic in the angular variable θ provided that the functions a_k and b_k are Hölder continuous.

Reduction to dimension two

Observation

It is enough to show the theorems when $n = 2$.

If $n > 2$, fix any plane P going through E and the origin. The theorems in the case $n = 2$ provide reconstruction in $B \cap P$. Repeating for all P gives the theorems.

Statement

We wish to prove:

Theorem

Suppose that $\Omega \subset \mathbb{R}^2$ is the unit disk, $E \subset \partial\Omega$ is a singleton and $f \in C(\bar{\Omega}, \mathbb{R})$. Then we can recover $f(0)$ from the broken ray transform $\mathcal{G}f$.

Long trajectories are spherically symmetric

- If there are very many reflections, the trajectory is almost spherically symmetric.
- When the number of reflections goes to infinity, the broken ray transform of f becomes an integral of f with a spherically symmetric weight.
- In this way we recover the integral

$$\mathcal{A}_0 a_0(z) = 2\int_z^1 \frac{a_0(r)}{\sqrt{1-(z/r)^2}}\mathrm{d}r,$$

where

$$a_0(r) = ⨍_{\partial B(0,r)} f\,\mathrm{d}\mathcal{H}^1$$

is the angular average of f.

The Abel transform

- The abel transform $\mathcal{A}_0$,

$$\mathcal{A}_0 g(z) = 2\int_z^1 \frac{g(r)}{\sqrt{1-(z/r)^2}}\mathrm{d}r,$$

  is injective.
- From $\mathcal{A}_0 a_0$ we can recover a_0, the angular average of f.
- Finally $f(0) = \lim_{r\to 0} a_0(r)$.

Statement

We wish to prove:

Theorem

Suppose that $\Omega \subset \mathbb{R}^2$ is the unit disk, $E \subset \partial\Omega$ is open and $f : \bar{\Omega} \to \mathbb{R}$ is uniformly quasianalytic in the angular variable. Then we can recover f everywhere from the broken ray transform $\mathcal{G}f$.

Fourier series

- Write the unknown function as a Fourier series in the angular variable:

$$f(r,\theta) = a_0(r) + \sum_{k=1}^{\infty} [a_k(r)\cos(k\theta) + b_k(r)\sin(k\theta)].$$

- Calculate the broken ray transform term by term. This involves the generalized Abel transform

$$\mathcal{A}_k g(z) = 2\int_z^1 T_k(z/r)\frac{g(r)}{\sqrt{1-(z/r)^2}}\mathrm{d}r,$$

where T_k are the Chebyshev polynomials.

Generalized Abel transform

- The generalized Abel transform is injective:

$$\mathcal{A}_k g(z) = 2\int_z^1 T_k(z/r)\frac{g(r)}{\sqrt{1-(z/r)^2}}\mathrm{d}r,$$
$$g(r) = -\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^1 T_k(z/r)\frac{\mathcal{A}_k g(z)}{z\sqrt{(z/r)^2-1}}\mathrm{d}z.$$

- The Radon transform in the plane can be conveniently written in terms of generalized Abel transforms of the Fourier components (Cormack 1963).

Rotation symmetry

- The integral of a function over a rotated trajectory equals the integral of a rotated function over the original trajectory.
- If $\mathcal{G}f = 0$, then also a slightly rotated versions of f have vanishing broken ray transform.
- Thus $\mathcal{G}f = 0 \Rightarrow \mathcal{G}\partial_{\mathrm{ang}}^n f = 0$ for all n.
- Using this and the regularity assumptions we get $\mathcal{A}_k a_k = 0$ and $\mathcal{A}_k b_k = 0$ for all k.
- This shows that $f = 0$ and concludes the proof.

Radon transform via Abel transforms

If

$$f(r,\theta)=\sum_{k\in\mathbb{Z}}a_k(r)e^{ik\theta},$$

then

$$\mathcal{R}f(r,\theta)=\sum_{k\in\mathbb{Z}}\mathcal{A}_k a_k(r)e^{ik\theta}.$$

The Radon transform $\mathcal{R}$ is parametrized so that

$$\mathcal{R}g(r,\vartheta)=\int_{L_{r,\vartheta}}g\,\mathrm{d}\mathcal{H}^1,$$

where $L_{r,\vartheta}=\{x\in\mathbb{R}^2 : x_1\cos\vartheta+x_2\sin\vartheta=r\}$.

Broken ray transform via Abel transforms

- We define the coefficient

$$S_k(\gamma) = \begin{cases} \dfrac{\sin(k(\kappa_\gamma - \iota_\gamma)/2)}{\sin(k\alpha_\gamma/2)} & \text{when } k\alpha_\gamma \notin 2\pi\mathbb{Z} \\ n_\gamma(-1)^{(n_\gamma+1)\frac{k\alpha_\gamma}{2\pi}+km_\gamma} & \text{when } k\alpha_\gamma \in 2\pi\mathbb{Z} \end{cases}$$

  for all $k = 0, 1, \ldots$ and $\gamma \in \Gamma_E$.
- If γ is symmetric w.r.t. to the angle zero, then

$$\mathcal{G}f(\gamma) = \text{length}(\gamma)^{-1} \sum_{k=0}^{\infty} S_k(\gamma)\mathcal{A}_k a_k(z_\gamma).$$

Uniformly quasianalytic functions

- $M = (M_n)_{n=0}^{\infty}$ is an increasing logarithmically convex sequence of strictly positive real numbers.
- A sequence $(a_k)_{k=1}^{\infty}$ is in $S^{\#}(M)$ if there is a constant $R > 0$ such that for each n

$$\sum_{k=0}^{\infty} k^{2n} |a_k|^2 \leq M_n R^n.$$

- The class $S^{\#}(M)$ is a *quasianalytic class of sequences* if M satisfies

$$\sum_{n=0}^{\infty} \sqrt{\frac{M_n}{M_{n+1}}} = \infty.$$

Uniformly quasianalytic functions

- A function $f : \mathbb{R} \to \mathbb{R}$ with a period 2π belongs to the class $C^{\#}(M)$ if it can be written as a uniformly convergent Fourier series as
$$f(x) = \sum_{k=0}^{\infty} (a_k \cos(kx) + b_k \sin(kx))$$
such that the sequences $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=1}^{\infty}$ are in the class $S^{\#}(M)$ and each $\partial^n_{\text{ang}} f$ is Dini-Lipschitz continuous.
- If the class $S^{\#}(M)$ is quasianalytic, then $C^{\#}(M)$ is a *quasianalytic class of functions*.

Uniformly quasianalytic functions

- A continuous function $f : \bar{D} \to \mathbb{R}$ is *uniformly quasianalytic in the angular variable* if for each $R \in (0, 1]$ the collection of functions $\{f(r, \cdot) : r \geq R\}$ belong to the same quasianalytic class.
- In dimensions three or higher, a function $f : \bar{B}^n \to \mathbb{R}$, $n \geq 3$, is uniformly quasianalytic in the angular variables if its restriction to any two dimensional plane intersecting the origin is uniformly quasianalytic in the sense defined in two dimensions.