JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

Spectral rigidity of the round Earth

CCIMI Seminar, University of Cambridge

Joonas Ilmavirta

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Based on joint work with

Maarten de Hoop and Vitaly Katsnelson (Rice)

Finnish Centre of Excellence in Inverse Modelling and Imaging

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CENTRES OF EXCELLENCE IN RESEARCH

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Conference announcement

A workshop “Inverse problems, PDE and geometry” will be organized in Jyväskylä, Finland, on August 20–23, 2018.

The workshop will focus on recent progress in the mathematical theory of inverse problems and related methods in PDEs, geometry and microlocal analysis.

Scientific committee: Yaroslav Kurylev (UCL), Mikko Salo (Jyväskylä), Gunther Uhlmann (Washington/HKUST)

Prelude

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- This is an inverse spectral problem. A hard one.
- There is a weaker version of the spectral problem: the spectral rigidity problem.
- Can we solve the simpler problem if we assume the Earth to be spherically symmetric?
- Yes!

Outline

The spectrum of free oscillations

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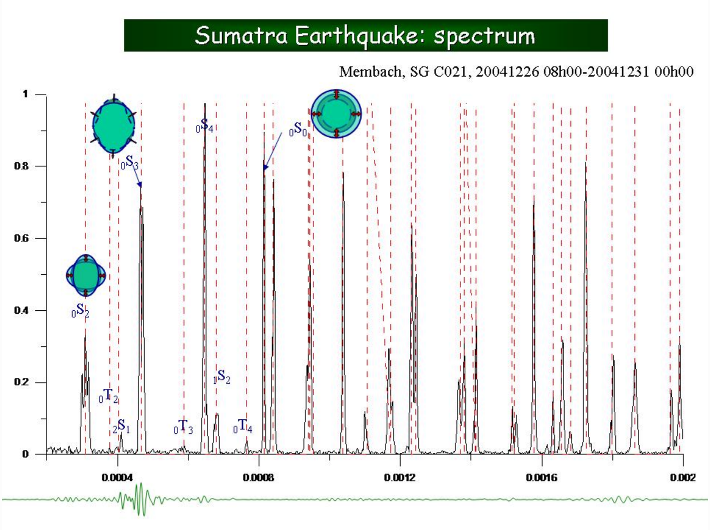
- The set of these frequencies (with multiplicity) is the spectrum of free oscillations.
- About 10 000 first frequencies are known.

The spectrum of free oscillations



Spectrum of free oscillations from an earthquake.

The spectrum of periodic orbits

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- Originally the length spectrum was just a mathematical tool, but it turns out it can be measured directly using deep earthquakes.

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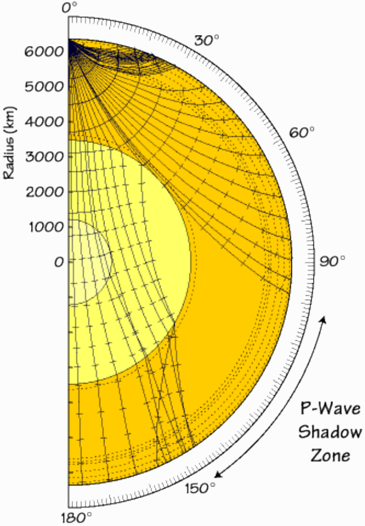

Seismic wave paths and the P-wave shadow zone. (Wikimedia Commons)

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We want to reconstruct the Earth in the natural Cartesian coordinates.

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- In practice, the Earth is the closed unit ball $M = \bar{B}(0,1) \subset \mathbb{R}^3$. The anisotropic sound speed is modeled with a Riemannian metric g on M.
- Physically, this corresponds to omitting S-waves and including only elliptic anisotropy.

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$$\Delta_g u(x) = c(x)^n \operatorname{div}(c(x)^{2-n} \nabla u(x)).$$

- The spectrum of free oscillations is the Neumann spectrum of the Laplace–Beltrami operator Δ_g.

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- The length spectrum of (M, g) is the set of all lengths of the periodic broken rays.

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- Proving this conjecture is difficult for two reasons:
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  2. The conjecture is false.

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- If $\phi\colon M \to M$ is a diffeomorphism, then (M, g) and $(M, \phi^* g)$ give the same spectrum.
- We can take any change of coordinates whatsoever and use it to distort the metric, but the spectrum stays the same.
- Physically: There are preferred and natural Cartesian coordinates. But the anisotropic model is not “sensitive to the underlying Euclidean geometry”, so the Cartesian coordinates cannot be recognized. It is impossible to find the metric (elliptically anisotropic sound speed) in Cartesian coordinates from spectral data.

Global uniqueness

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Let g_s be family of Riemannian metrics on a manifold M with boundary, depending on a parameter $s \in (-\varepsilon, \varepsilon)$. If they all give the same spectrum, are there a diffeomorphisms $\phi_s \colon M \to M$ so that $g_0 = \phi_s^* g_s$?

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  - Some more general manifolds: Paternain–Salo–Uhlmann 2015.
- We have adapted similar ideas of proof to manifolds with boundary.

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Spherically symmetric manifolds

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- If g is a rotation invariant Riemannian metric on M, there is a radial (more complicated if $n = 2$) diffeomorphism $\phi\colon M \to M$ so that $\phi^* g$ is radially conformally Euclidean.

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- If g is a rotation invariant Riemannian metric on M, there is a radial (more complicated if $n = 2$) diffeomorphism $\phi\colon M \to M$ so that $\phi^* g$ is radially conformally Euclidean.
- The Earth is spherically symmetric to a good approximation, but the best (elliptically anisotropic) radial model might not be conformally Euclidean. After a radial change of coordinates the metric becomes conformal — and Cartesian coordinates are lost.

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$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r}{c(r)}\right) > 0$$

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- The manifold is non-trapping and has strictly convex boundary.

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- In addition, the shear wave speed vanishes in the liquid outer core.
- Apart from these problems (jumps and liquid) both shear and pressure wave speeds do satisfy the Herglotz condition everywhere.

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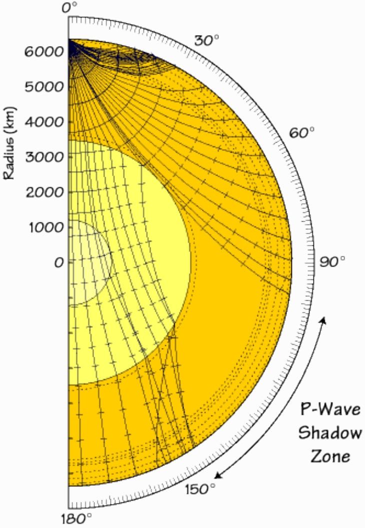

The Herglotz condition is satisfied: ray paths curve outwards. (Wikimedia Commons)

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Let M be the closed unit ball in $\mathbb{R}^3$. Let $c_s(r)$ be a family of radial sound speeds depending C^∞-smoothly on both $s \in (-\varepsilon, \varepsilon)$ and $r \in [0, 1]$. Assume each c_s satisfies the Herglotz condition and a generic geometrical condition.

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Let M be the closed unit ball in $\mathbb{R}^3$. Let g_s be a family of rotation invariant metrics depending C^∞-smoothly on $s \in (-\varepsilon, \varepsilon)$. Suppose each g_s is non-trapping with strictly convex boundary and assume a generic geometrical condition.

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If the spectra of the Laplace–Beltrami operators Δ_{g_s} are all equal, then there is a family of radial diffeomorphisms $\phi_s \colon M \to M$ so that $\phi_s^* g_s = g_0$ for all s. That is, the manifolds (M, g_s) are isometric.

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Ideas behind the proof

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Lemma (Trace formula)

Let $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the positive eigenvalues of the Laplace–Beltrami operator. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \sum_{k=0}^{\infty} \cos\left(\sqrt{\lambda_k} \cdot t\right).$$

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The function $f(t)$ *is singular precisely at the length spectrum.*

In particular, the spectrum determines the length spectrum.

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Similar “trace formulas” and related results are known on closed manifolds (eg. Duistermaat–Guillemin 1975) and a weaker version on some manifolds with boundary (eg. Guillemin–Melrose 1979).

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Corollary

Spectral rigidity follows from length spectral rigidity.

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In particular, if the length spectrum does not depend on s, then $\frac{\mathrm{d}}{\mathrm{d}s}c_s^{-2}$ integrates to zero over (almost) all periodic broken rays.

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Remark: No proof works without spherical symmetry.

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3. Linearized length spectral data is periodic broken ray transform data.
4. The periodic broken ray transform can be inverted explicitly for radial functions.

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If the length of the interval is L, the eigenvalues are

$$\lambda_k = \left(\frac{k\pi}{L}\right)^2, \quad k = 0, 1, 2, \ldots$$

Suppose $L = \frac{1}{2}$ and we have measured the numbers $0, 4\pi^2, 16\pi^2, \ldots$

A numerical example of the trace formula

Example: We want to find the length of an interval, given its Neumann spectrum.

If the length of the interval is L, the eigenvalues are

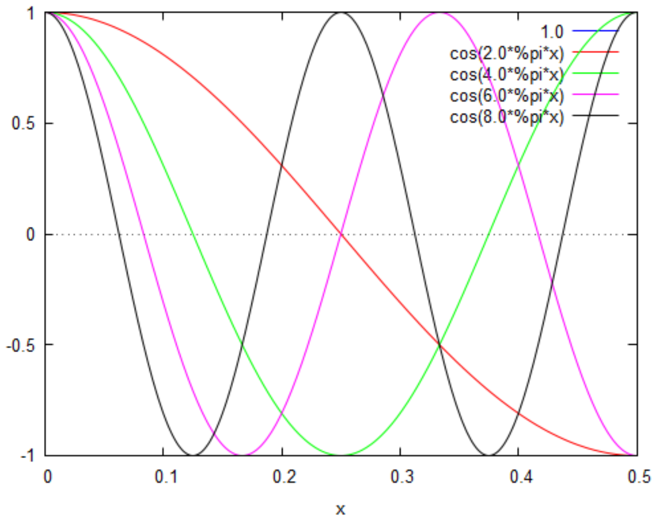
$$\lambda_k = \left(\frac{k\pi}{L}\right)^2, \quad k = 0, 1, 2, \ldots$$

Suppose $L = \frac{1}{2}$ and we have measured the numbers $0, 4\pi^2, 16\pi^2, \ldots$

We compute and plot the trace function

$$f(t) = \sum_{k=0}^{\infty} \cos(\sqrt{\lambda_k} \cdot t).$$

A numerical example of the trace formula



Eigenfunctions for $k = 0, 1, 2, 3, 4$.

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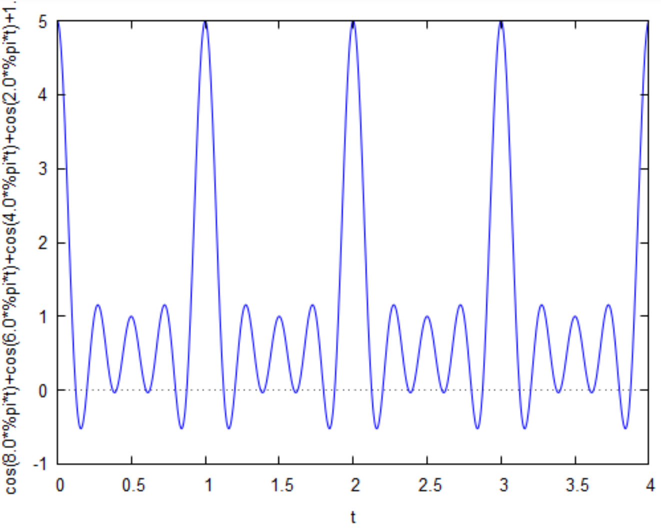

Trace function computed from $k = 0, 1, 2, 3, 4$.

Outline

Elliptic and general elastic anisotropy

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  - General anisotropy corresponds to a Finsler manifold (a manifold with a Finsler metric).
  - Riemannian manifolds are a very special subclass of Finsler manifolds.
- A material is isotropic if sound speed is independent of direction. This can be modeled by a conformally Euclidean metric.

Pressure and shear waves

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- To model elastic waves in general anisotropy, one needs a manifold with two Finsler metrics, one for pressure and one for shear waves.
- In fact, the shear wave speed might not even by a Finsler metric in the traditional sense.

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- A fully anisotropic model can *never* be reconstructed from boundary measurements uniquely. The data is always invariant under changes of coordinates.
- The best one can hope for is reconstruction up to changes of coordinates.

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- Isotropic P-wave speed. — Conformally Euclidean metric.
- Spherical symmetry.
- Reconstruction possible in the natural Cartesian coordinates. — No gauge freedom.

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Theorem (de Hoop–I., 2017)

Let M be a rotation symmetric non-trapping manifold with a piecewise $C^{1,1}$ metric and strictly convex boundary. Then the geodesic X-ray transform is injective on $L^2(M)$.

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Earlier similar results:

- The X-ray transform (Radon et al.): Euclidean metric (c is constant).
- Mukhometov, 1977: Smooth simple metrics (simplicity is stronger than Herglotz).
- Sharafutdinov, 1997: C^∞ metrics and C^∞ functions.

Periodic broken ray transforms

Periodic broken ray transforms

Theorem (de Hoop–I., 2017)

Let M be a rotation symmetric non-trapping manifold with a $C^{1,1}$ metric and strictly convex boundary and dimension at least three. Assume that there are not too many conjugate points at the boundary. The integrals of a function $f \in L^p(M)$, $p > 3$, over all periodic broken rays determines the even part of the function.

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Tools used:

- Planar average ray transform.
- Abel transform.
- Funk transform.
- Fourier series.

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