JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

Finsler geometry from the elastic wave equation

BIRS workshop

Probing the Earth and the Universe with Microlocal Analysis

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April 16, 2019

Based on joint work with

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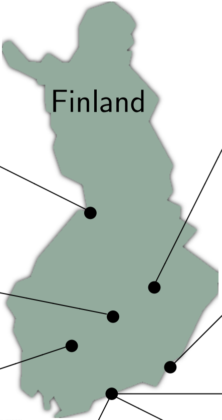
Finnish Centre of Excellence in Inverse Modelling and Imaging

2018-2025

CENTRES OF EXCELLENCE IN RESEARCH



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FINNISH METEOROLOGICAL INSTITUTE

Aalto University

Conference announcement

The annual Finnish inverse problems conference “Inverese Days” will be organized in Jyväskylä 16–18 December, 2019.

`http://r.jyu.fi/yVK`

(`https://www.jyu.fi/science/en/maths/research/inverse-problems/id2019/`)

All kinds of inverse problems in all fields are welcome!

Goals

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- Overview of fully anisotropic linear elasticity.

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- How geometrization leads naturally to Finsler geometry.

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- How geometrization leads naturally to Finsler geometry.
- Examples of geometric inverse problems in the Finsler setting.

Outline

The stiffness tensor

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- The tensor is very symmetric ($c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$) and positive ($\sum_{i,j,k,l} c_{ijkl}\alpha_i\beta_j\beta_k\alpha_l \gtrsim |\alpha|^2 |\beta|^2$).

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- We will also encounter the density normalized stiffness tensor $a_{ijkl}(x) = c_{ijkl}(x)/\rho(x)$.

The elastic wave equation

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- Using Newton's second law with a restoring force given by Hooke's law leads to the elastic wave equation (EWE)

$$\partial_j[c_{ijkl}(x)\partial_k u_l(x,t)] - \rho(x)\partial_t^2 u_i(x,t) = 0,$$

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- Elastic waves arising from earthquakes (or marsquakes!) satisfy this equation away from the focus of the event.

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- The principal symbol of the EWE is $\Gamma(x,\xi) - \omega^2 I$, where $\xi = \omega p$.

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- Decomposition to polarizations only works on the level of singularities. The individual polarizations do not satisfy PDEs.

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- The admissible slowness vectors are on the slowness surface given by the equation
$$\det(\Gamma(x, p) - I) = 0.$$

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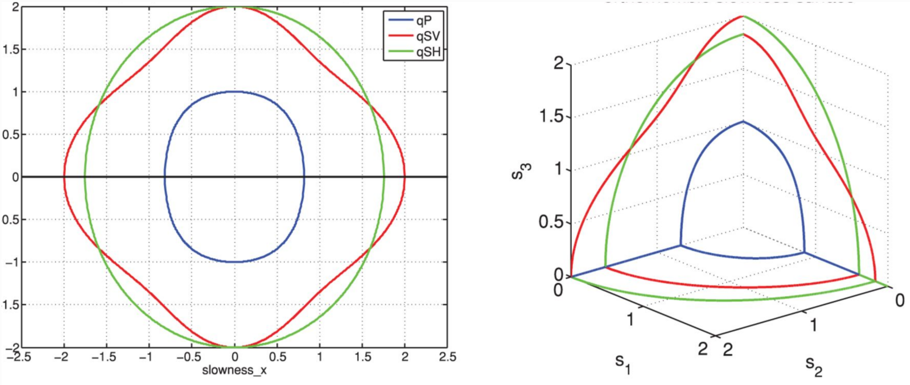

The slowness surface. Smaller slowness $\iff$ faster wave.

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- The qS branch of the slowness surface might not be convex.
- We will focus on qP waves.

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- A more geometric formulation: Given some boundary data, find the slowness surface at every point.
- To solve the physical problem, it remains to uniquely determine the tensor a from the slowness surface or a branch thereof.

Outline

1. The elastic wave equation
2. Finsler geometry
   - Finsler manifolds
   - Elastic Finsler manifolds
   - Properties on the fiber
   - Local Riemannian metric
   - Inverse problems
3. Examples of inverse problems in Finsler geometry

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- Lengths of curves are defined in the usual way using the (Minkowski) norm on every tangent space.

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- The qP singularities follow the Hamiltonian flow of $\lambda\colon T^*M \to \mathbb{R}$.

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- We have described Finsler geometry on the cotangent side.

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- We have found a Finsler manifold $(\mathbb{R}^3, F)$ whose geodesic flow corresponds to the propagation of qP singularities.

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- In elastic Finsler geometry the distance between two points $x, y \in \mathbb{R}^3$ is the shortest time in which an elastic wave can go from x to y.
- Declaring travel time as distance would have defined the same geometry, but in a more implicit manner.

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- The different branches of the slowness surface are not algebraically independent.

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- If there is a preferred direction (given e.g. by a geodesic or normals of a hypersurface), then there is a natural Riemannian metric on (M, F). Connections and other objects are most convenient in this Riemannian geometry.

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- To find the whole Finsler function, one needs access to all directions on TM, whereas in Riemannian geometry it often suffices to reach every base point.
- Finsler structures arising from elasticity resemble Riemannian metrics in a useful way: they are fiberwise real-analytic. Therefore access to an open subset of every fiber is enough.

Inverse problems

- A typical inverse problem in Finsler geometry would be: Given some boundary data, find (M, F).
- This is the same problem as finding the slowness surface everywhere — modulo diffeomorphisms.
- To find the whole Finsler function, one needs access to all directions on TM, whereas in Riemannian geometry it often suffices to reach every base point.
- Finsler structures arising from elasticity resemble Riemannian metrics in a useful way: they are fiberwise real-analytic. Therefore access to an open subset of every fiber is enough.
- Whether the elastic problem has the diffeomorphism gauge freedom is another question; cf. András's talk on Monday.

Outline

1. The elastic wave equation
2. Finsler geometry
3. Examples of inverse problems in Finsler geometry
   - Herglotz (Mönkkönen)
   - Dix (de Hoop, Lassas)
   - Distance function (de Hoop, Lassas, Saksala)
   - Scattering data (de Hoop, Lassas, Saksala)

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- This is not true for a spherically symmetric Finsler manifold.
- There is still a Herglotz condition but it looks different.
- Linearized travel time data leads to X-ray tomography. If the stiffness tensor c is known but ρ unknown, the variations are conformal.

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- In some measurement set $U \subset M$ one can see spheres with any center. The data consists of oriented surfaces with radii.
- One can follow the geodesics backwards and find the metric on a neighborhood of the lift.
- With fiberwise analyticity this information can be globalized to give the universal cover of (M, F).

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- Teemu will tell more on Friday.

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- This broken scattering relation can see much more of TM, but the trapped set is still invisible.
- Global uniqueness is doable (done) with added assumptions: reversibility and foliation.
- Almost no assumptions are needed in the Riemannian case (Kurylev–Lassas–Uhlmann, 2010).

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- making it all work in real life.

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