Spectral rigidity of the round Earth

Applied Inverse Problems

Joonas Ilmavirta (University of Jyväskylä)
with Maarten de Hoop and Vitaly Katsnelson (Rice University)

Slides and papers will appear at http://users.jyu.fi/~jojapeil

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- This is an inverse spectral problem. A hard one.
- There is a weaker version of the spectral problem: the spectral rigidity problem.
- Can we solve the simpler problem if we assume the Earth to be spherically symmetric?
- Yes!

Outline

1. Seismic spectral data
   - The spectrum of free oscillations
   - The spectrum of periodic orbits
   - The goal
2. Spectra of a manifold with boundary
3. Different forms of uniqueness
4. Spherical symmetry
5. The main results
6. Anisotropy and geometry
7. Ray transforms in low regularity

The spectrum of free oscillations

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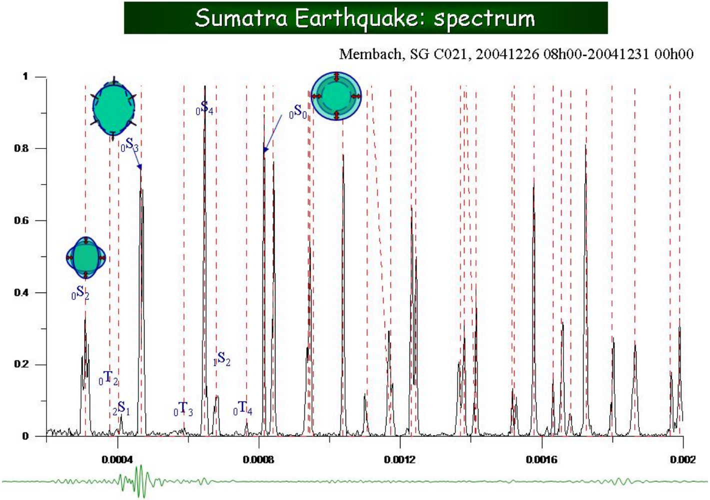
- The set of these frequencies is the spectrum of free oscillations.
- About 10 000 first frequencies are known.

The spectrum of free oscillations



Spectrum of free oscillations from an earthquake.

The spectrum of periodic orbits

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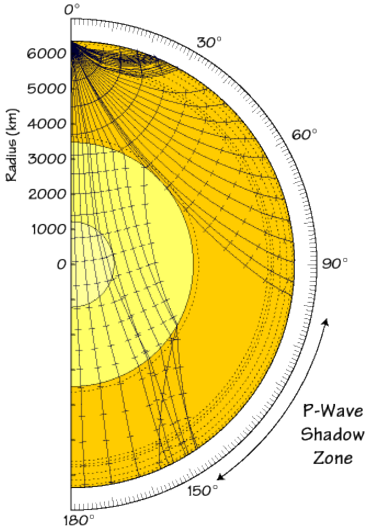
- Some of the wave paths are periodic. Every periodic wave path has a length (in time).
- The set of all lengths of periodic seismic wave paths is the "length spectrum" of the Earth.
- Originally the length spectrum was just a mathematical tool, but it turns out it can be measured directly using deep earthquakes.

The spectrum of periodic orbits



Seismic wave paths and the P-wave shadow zone. (Wikimedia Commons)

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We want to reconstruct the Earth in the natural Cartesian coordinates.

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1. Seismic spectral data
2. Spectra of a manifold with boundary
   - Manifolds with boundary
   - The spectrum of the Laplacian
   - The length spectrum
3. Different forms of uniqueness
4. Spherical symmetry
5. The main results
6. Anisotropy and geometry
7. Ray transforms in low regularity

Manifolds with boundary

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- We model the Earth as a Riemannian manifold with boundary.
- In practice, the Earth is the closed unit ball $M = \bar{B}(0,1) \subset \mathbb{R}^3$. The anisotropic sound speed is modeled with a Riemannian metric g on M.
- Physically, this corresponds to omitting S-waves and including only elliptic anisotropy.

The spectrum of the Laplacian

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- If the sound speed is isotropic, then $g = c^{-2}e$ and the Laplace–Beltrami operator in dimension n is

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- The spectrum of free oscillations is the Neumann spectrum of the Laplace–Beltrami operator Δ_g.

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- Seismic waves reflect at the surface, so they are in fact billiard trajectories or broken rays.
- The length spectrum of (M, g) is the set of all lengths of the periodic broken rays.

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2. Spectra of a manifold with boundary
3. Different forms of uniqueness
   - Difficulties
   - Diffeomorphisms and coordinates
   - Global uniqueness
   - Local uniqueness
   - Spectral rigidity
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- Proving this conjecture is difficult for two reasons:
    1. The required tools do not yet exist on general manifolds with boundary.
    2. The conjecture is false.

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- If $\phi \colon M \to M$ is a diffeomorphism, then (M, g) and $(M, \phi^* g)$ give the same spectrum.
- We can take any change of coordinates whatsoever and use it to distort the metric, but the spectrum stays the same.
- Physically: There are preferred and natural Cartesian coordinates. But the anisotropic model is not "sensitive to the underlying Euclidean geometry", so the Cartesian coordinates cannot be recognized. It is impossible to find the metric (anisotropic sound speed) in Cartesian coordinates from spectral data.

Global uniqueness

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Let g_1 and g_2 be two Riemannian metrics on a manifold M with boundary. If they give the same spectrum, is there a diffeomorphism $\phi\colon M \to M$ so that $g_1 = \phi^* g_2$?

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This is too hard.

Local uniqueness

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Let g_1 and g_2 be two Riemannian metrics on a manifold M with boundary. Suppose g_1 is very close to g_2. If they give the same spectrum, is there a diffeomorphism $\phi \colon M \to M$ so that $g_1 = \phi^* g_2$?

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This is still too hard.

Spectral rigidity

Problem

Let g_s *be family of Riemannian metrics on a manifold* M *with boundary, depending on a parameter* $s \in (-\varepsilon, \varepsilon)$*. If they all give the same spectrum, are there a diffeomorphisms* $\phi_s \colon M \to M$ *so that* $g_0 = \phi_s^* g_s$*?*

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In other words, are isospectral deformations necessarily trivial?

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This is within reach!

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  - Negatively curved surfaces: Guillemin–Kazhdan 1980.
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  - Some more general manifolds: Paternain–Salo–Uhlmann 2015.
- We have adapted similar ideas of proof to manifolds with boundary.

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4. Spherical symmetry
   - Spherically symmetric manifolds
   - The Herglotz condition
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Spherically symmetric manifolds

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- If g is a rotation invariant Riemannian metric on M, there is a radial (more complicated if $n = 2$) diffeomorphism $\phi \colon M \to M$ so that $\phi^* g$ is radially conformally Euclidean.
- The Earth is spherically symmetric to a good approximation, but the best (elliptically anisotropic) radial model might not be conformally Euclidean. After a radial change of coordinates the metric becomes conformal — and Cartesian coordinates are lost.

The Herglotz condition

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A radial sound speed $c(r)$ satisfies the Herglotz condition if

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Equivalent formulations:

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- The manifold is non-trapping and has strictly convex boundary.

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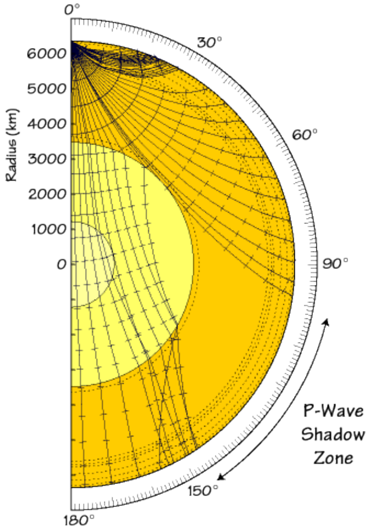
- In addition, the shear wave speed vanishes in the liquid outer core.
- Apart from these problems (jumps and liquid) both shear and pressure wave speeds do satisfy the Herglotz condition everywhere.

The Herglotz condition



Some P-waves are trapped in the outer core. (Wikimedia Commons)

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   - Length spectral rigidity
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Spectral rigidity

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Let M be the closed unit ball in $\mathbb{R}^3$. Let $c_s(r)$ be a family of radial sound speeds depending C^∞-smoothly on both $s \in (-\varepsilon, \varepsilon)$ and $r \in [0, 1]$. Assume each c_s satisfies the Herglotz condition and a generic geometrical condition.

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This simple model of the round Earth is spectrally rigid!

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Let M *be the closed unit ball in* $\mathbb{R}^3$*. Let* g_s *be a family of rotation invariant metrics depending* C^∞*-smoothly on* $s \in (-\varepsilon, \varepsilon)$*. Suppose each* g_s *is non-trapping with strictly convex boundary and assume a generic geometrical condition.*

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If the spectra of the Laplace–Beltrami operators Δ_{g_s} are all equal, then there is a family of radial diffeomorphisms $\phi_s \colon M \to M$ so that $\phi_s^* g_s = g_0$ for all s. That is, the manifolds (M, g_s) are isometric.

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If the length spectra of the manifolds (M, g_s) are all equal, then there is a family of radial (or more general if $n = 2$) diffeomorphisms $\phi_s \colon M \to M$ so that $\phi_s^* g_s = g_0$ for all s. That is, the manifolds (M, g_s) are isometric.

Ideas behind the proof

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Lemma (Trace formula)

Let $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ *be the positive eigenvalues of the Laplace–Beltrami operator. Define a function* $f\colon \mathbb{R} \to \mathbb{R}$ *by*

$$f(t) = \sum_{k=0}^{\infty} \cos\left(\sqrt{\lambda_k} \cdot t\right).$$

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In particular, the spectrum determines the length spectrum.

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Similar "trace formulas" and related results are known on closed manifolds (eg. Duistermaat–Guillemin 1975) and a weaker version on some manifolds with boundary (eg. Guillemin–Melrose 1979).

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Corollary

Spectral rigidity follows from length spectral rigidity.

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In particular, if the length spectrum does not depend on s*, then* $\frac{\mathrm{d}}{\mathrm{d}s}c_s^{-2}$ *integrates to zero over (almost) all periodic broken rays.*

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Remark: No proof works without spherical symmetry.

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6. Anisotropy and geometry
   - Elliptic and general elastic anisotropy
   - Pressure and shear waves
   - Anisotropy and coordinates
   - Our model
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  - Riemannian manifolds are a very special subclass of Finsler manifolds.
- A material is isotropic if sound speed is independent of direction. This can be modeled by a conformally Euclidean metric.

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- To model elastic waves in general anisotropy, one needs a manifold with two Finsler metrics, one for pressure and one for shear waves.
- In fact, the shear wave speed might not even by a Finsler metric in the traditional sense.

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- A fully anisotropic model can *never* be reconstructed from boundary measurements uniquely. The data is always invariant under changes of coordinates.
- The best one can hope for is reconstruction up to changes of coordinates.

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- Reconstruction possible in the natural Cartesian coordinates. — No gauge freedom.

Outline

1. Seismic spectral data
2. Spectra of a manifold with boundary
3. Different forms of uniqueness
4. Spherical symmetry
5. The main results
6. Anisotropy and geometry
7. Ray transforms in low regularity
   - X-ray transforms
   - Periodic broken ray transforms

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Earlier similar results:

- The X-ray transform (Radon et al.): Euclidean metric (c is constant).
- Mukhometov, 1977: Smooth simple metrics (simplicity is stronger than Herglotz).
- Sharafutdinov, 1997: C^∞ metrics and C^∞ functions.

Periodic broken ray transforms

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Let M be a rotation symmetric non-trapping manifold with a $C^{1,1}$ metric and strictly convex boundary and dimension at least three. Assume that there are not too many conjugate points at the boundary. The integrals of a function $f \in L^p(M)$, $p > 3$, over all periodic broken rays determines the even part of the function.

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Very little can be recovered of the odd part.

Tools used:

- Planar average ray transform.
- Abel transform.
- Funk transform.
- Fourier series.

End

Slides and papers will appear at http://users.jyu.fi/~jojapeil.

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Thank you.