JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

Inverse problems in elastic Finsler geometry

Applied Inverse Problems minisymposium
Recent advances in geometric inverse problems

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July 9, 2019

Based on joint work with
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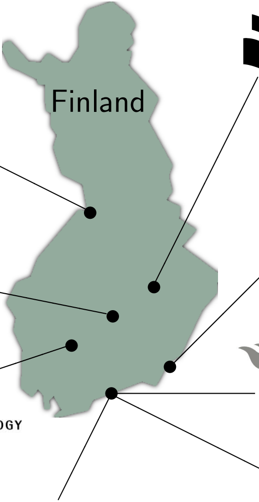
Finnish Centre of Excellence in Inverse Modelling and Imaging

2018-2025

CENTRES OF EXCELLENCE IN RESEARCH



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Aalto University

Conference announcement

The annual Finnish inverse problems conference "Inverse Days" will be organized in Jyväskylä 16–18 December, 2019.

http://r.jyu.fi/yVK

(https://www.jyu.fi/science/en/maths/research/inverse-problems/id2019/)

Registration opens this week!

Goals

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- How geometrization leads naturally to Finsler geometry.

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- How geometrization leads naturally to Finsler geometry.
- Examples of geometric inverse problems in the Finsler setting.

Outline

The stiffness tensor

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- The tensor is very symmetric ($c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$) and quite positive ($c_{ijkl}\alpha_i\beta_j\beta_k\alpha_l \gtrsim |\alpha|^2 |\beta|^2$).

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- We will also encounter the density normalized stiffness tensor $a_{ijkl}(x) = c_{ijkl}(x)/\rho(x)$.

The elastic wave equation

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- Using Newton's second law with a restoring force given by Hooke's law leads to the elastic wave equation (EWE)

$$\partial_j[c_{ijkl}(x)\partial_k u_l(x,t)] - \rho(x)\partial_t^2 u_i(x,t) = 0,$$

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- Elastic waves arising from earthquakes (or marsquakes!) satisfy this equation away from the focus of the event to great accuracy.

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- The principal symbol of the EWE is $\Gamma(x,\xi) - \omega^2 I$, where $\xi = \omega p$.

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- Polarization vectors are eigenvectors of the Christoffel matrix Γ, so they are orthogonal.
- Decomposition to polarizations only works on the level of singularities. The individual polarizations do not satisfy PDEs.

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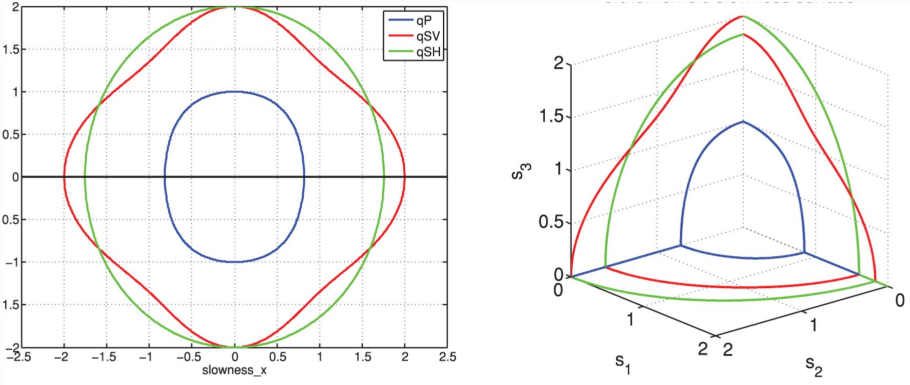
- The admissible slowness vectors are on the slowness surface given by the equation

$$\det(\Gamma(x,p) - I) = 0.$$

Singularities and the slowness surface



The slowness surface. Smaller slowness $\iff$ faster wave.

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- A more geometric formulation: Given some boundary data, find the slowness surface at every point.
- To solve the physical problem, it remains to uniquely determine the tensor a from the slowness surface or a branch thereof.

Outline

1. The elastic wave equation
2. Finsler geometry
   - Finsler manifolds
   - Elastic Finsler manifolds
   - Properties on the fiber
   - Inverse problems
3. Examples of inverse problems in Finsler geometry

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- Lengths of curves are defined in the usual way using the (Minkowski) norm on every tangent space.

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- The qP singularities follow the Hamiltonian flow of $\lambda \colon T^*M \to \mathbb{R}$.

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- Slowness is a covector and the corresponding vector is the group velocity.

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- In elastic Finsler geometry the distance between two points $x, y \in \mathbb{R}^3$ is the shortest time in which an elastic wave can go from x to y.
- Declaring travel time as distance would have defined the same geometry, but in a more implicit manner.

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  3. The flow on $T^*\mathbb{R}^3$ is still given by the Hamiltonian corresponding to an eigenvalue of Γ, but it can be non-convex. The metric on $T\mathbb{R}^3$ is multiple-valued or its geodesic flow does not correspond to singularities.

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- Whether the elastic problem has the diffeomorphism gauge freedom is another question.

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1. The elastic wave equation
2. Finsler geometry
3. Examples of inverse problems in Finsler geometry
   - Herglotz (Mönkkönen)
   - Dix (de Hoop, Lassas)
   - Distance function (de Hoop, Lassas, Saksala)
   - Scattering data (de Hoop, Lassas, Saksala)

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- There is still a Herglotz condition but it looks different.
- Linearized travel time data leads to X-ray tomography. If the stiffness tensor c is known but ρ unknown, the variations are conformal.

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- With fiberwise analyticity this information can be globalized to give the universal cover of (M, F).
- The "directionality" of Finsler geometry is a major complication in comparison to the Riemannian version (de Hoop–Holman–Iversen–Lassas–Ursin, 2015).

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- One can reconstruct M and F on the good set $G \subset TM$, but not outside it.

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- If F is fiberwise real analytic (elasticity or Riemann!), then F is determined uniquely.

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- This broken scattering relation can see much more of TM, but the trapped set is still invisible.
- Global uniqueness is doable (done) with added assumptions: reversibility and foliation.
- Almost no assumptions are needed in the Riemannian case (Kurylev–Lassas–Uhlmann, 2010).

Goals

- Overview of fully anisotropic linear elasticity.
- How geometrization leads naturally to Finsler geometry.
- Examples of geometric inverse problems in the Finsler setting.

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