JYVÄSKYLÄN YLIOPISTO
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Geophysics and algebraic geometry

NCSU Geometry and Topology Seminar

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Based on joint work with

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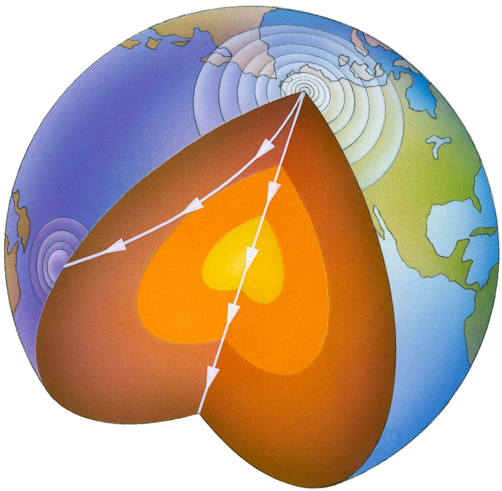
The question



How to see the interior of the Earth via seismic rays?

Outline

Elastic wave equation

Quantities:

- Displacement $u(t, x) \in \mathbb{R}^n$.
- Density $\rho(x) \in \mathbb{R}$.
- Stiffness tensor $c_{ijkl}(x) \in \mathbb{R}^{n^4}$.

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Properties:

- $\rho > 0$.
- $c_{ijkl} = c_{klij} = c_{jikl}$.
- $\sum_{i,j,k,l} c_{ijkl} A_{ij} A_{kl} > 0$ whenever $A = A^T \neq 0$.

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Equation of motion (EWE):
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Compare: Newton ($F = m\ddot{x}$) and Hooke ($F = -kx$).

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If $u = Ae^{i\omega(t-p\cdot x)}$, then the EWE becomes

$$\rho\omega^2[-I + \Gamma(p)]A = 0,$$

where

$$\Gamma_{il}(p) = \sum_{j,k} \rho^{-1} c_{ijkl} p_j p_k$$

is the Christoffel matrix.

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This describes where the singularities (point particles instead of waves) can be but not yet how they can move.

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- λ_m defines a Hamiltonian on $T^*\Omega$ and the singularities corresponding to the mth fastest eigenvalue follow the Hamiltonian flow.
- $\lambda_1^{1/2}$ defines a norm on $T^*\Omega$.
- The dual norm $F = (\lambda_1^{1/2})^*$ on $T\Omega$ is a Finsler norm.
- The singularities follow the geodesics of the Finsler geometry given by F.

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The propagation of singularities only depends on the reduced stiffness tensor $a = \rho^{-1}c$.

Slowness polynomial and slowness surface

A reduced stiffness tensor a_{ijkl} defines

- a Christoffel matrix $\Gamma_a(p)$ and
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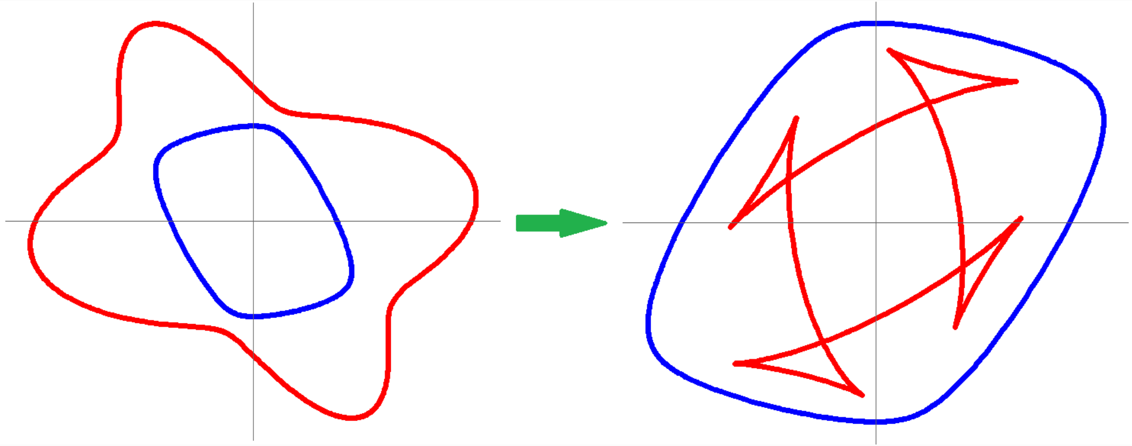
The set where singularities are possible is the slowness surface

$$\Sigma_a = \{p \in \mathbb{R}^n; P_a(p) = 0\}.$$

Knowing the slowness polynomial $\iff$ knowing the slowness surface.

Slowness polynomial and slowness surface

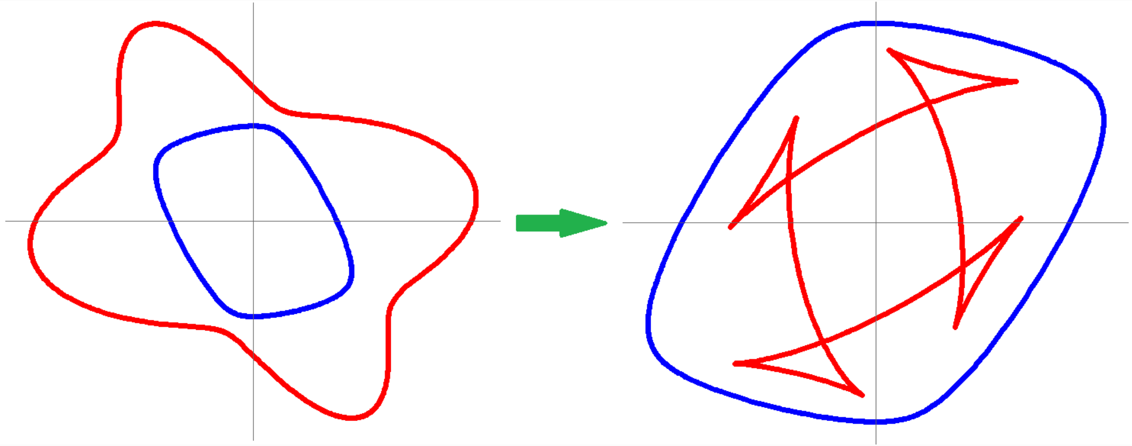


A slowness surface in 2D with its two branches, and the corresponding two Finsler norms.

The quasi pressure (qP) polarization behaves well.

Anisotropy $\iff$ dependence on direction $\iff$ not circles.

Slowness polynomial and slowness surface



Left: The set (slowness surface) of cotangent vectors, momenta, or phase velocities in $T_x^*\Omega$.
Right: The set of tangent vectors, velocities, group velocities in $T_x\Omega$.
Duality between microlocal analysis and algebra on the left and geometry on the right.

Geometrization of an analytic problem

Original inverse problem

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Remarks:

- Geometric inverse problems like this can be solved for qP geometries.
- Riemannian geometry is not enough; it can only handle a tiny subclass of physically valid and interesting stiffness tensors.
- Knowing the metric is the same as knowing the (co)sphere bundle: (M, g) or (M, F) $\iff$ (M, SM) $\iff$ (M, S^*M).
- The cospheres of the Finsler geometry are the qP branches of the slowness surfaces.

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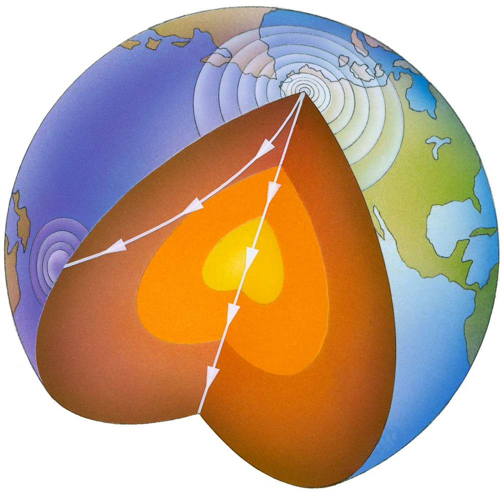

Rays follow geodesics and tell about the interior structure.

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The study of the geometry of varieties is a part of algebraic geometry.

Algebraic variety

Given any set F of functions $f\colon \mathbb{R}^n \to \mathbb{R}$, we can define a closure for all $A \subset \mathbb{R}^n$:

$$\mathrm{cl}_F(A) = \{x \in \mathbb{R}^n; \forall f \in F : f|_A = 0 \implies f(x) = 0\}.$$

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A variety is the same as a Zariski-closed set.

Generic irreducibility

Definition

A variety $V \subset \mathbb{R}^n$ is reducible if it can be written as the union of two varieties in a non-trivial way.

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This is not true for all a but only generically.

Generic irreducibility

Corollary (de Hoop, Ilmavirta, Lassas, Várilly-Alvarado)

When the slowness surface Σ_a is irreducible, any (Euclidean) relatively open subset determines the whole slowness surface.
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It suffices to measure the well-behaved qP branch!

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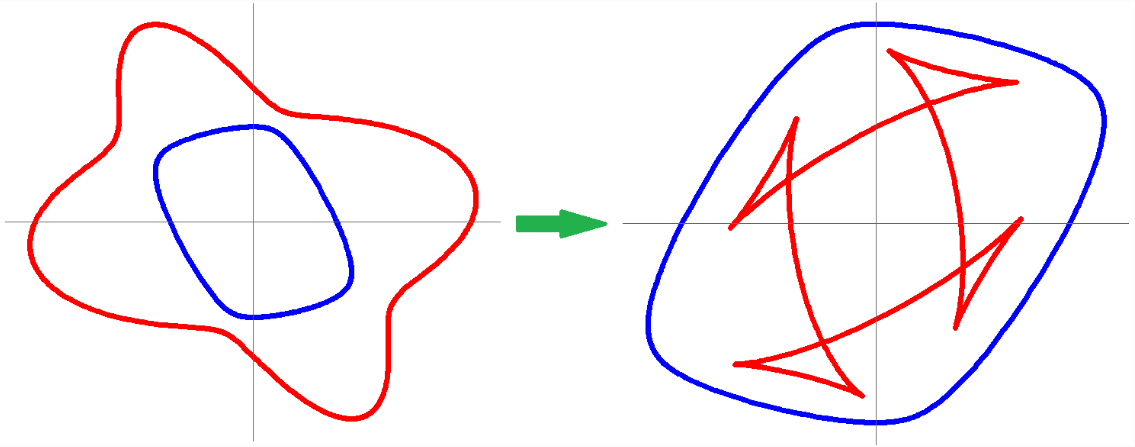

Any small part of the well-behaved quasi pressure branch determines the whole thing via Zariski closure.

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- It takes the full power of scheme theory to prove that the set of stiffness tensors a for which the slowness polynomial P_a is irreducible is open in the Zariski topology.
- It takes a single concrete example to show that that set is not empty.

Generically unique reduced stiffness tensor

Theorem (de Hoop–Ilmavirta–Lassas–Várilly-Alvarado)

Let $n \in \{2, 3\}$. There is an open and dense subset W of stiffness tensors a so that the map $W \ni a \to P_a$ is injective.

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Corollary (de Hoop–Ilmavirta–Lassas–Várilly-Alvarado)

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We may think of the real or complex slowness surface, a subset in $\mathbb{R}^n$ or $\mathbb{C}^n$.
The slowness polynomial stays the same.

Singularity

Theorem (Ilmavirta)

Let $n \notin \{1, 2, 4, 8\}$. Then for all stiffness tensors $a > 0$ the complex slowness surface is singular.

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Theorem (Ilmavirta)

Let $n = 2$. Then the real and complex slowness surface is generically smooth. There is a simple test for singularity.

The case $n = 1$ is uninteresting and the cases $n \in \{4, 8\}$ are open.
The qP branch can still be smooth — and it often is.
This is not typical behaviour of a family of varieties: slowness surfaces are special.

Characterization of slowness polynomials

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We know this to be true in all dimensions, but we do not know the polynomials.

Outline

1. Inverse problems in elasticity
2. Geometry of slowness surfaces
3. Coordinate gauge
   - Coordinate gauge in geometric inverse problems
   - Degeometrization

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Only the isometry class of the manifold matters, so in a coordinate representation there is a gauge freedom of diffeomorphisms.

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Question

Let a and b be two different stiffness tensor fields on a domain $\Omega \subset \mathbb{R}^n$ and $\phi \colon \Omega \to \Omega$ a diffeomorphism fixing the boundary. Is it possible that $F_a^{qP} = \phi^* F_b^{qP}$ — i.e., that (Ω, F_a^{qP}) and (Ω, F_b^{qP}) are isometric?

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The pullback $\phi^* a$ may also lose symmetries.

Degeometrization

The solution to the geometrized problem on a Finsler manifold has the coordinate gauge freedom. But how about the original problem?

Question

Let a and b be two different stiffness tensor fields on a domain $\Omega \subset \mathbb{R}^n$ and $\phi \colon \Omega \to \Omega$ a diffeomorphism fixing the boundary. Is it possible that $F_a^{qP} = \phi^* F_b^{qP}$ — i.e., that (Ω, F_a^{qP}) and (Ω, F_b^{qP}) are isometric?

It turns out that the answer depends heavily on the symmetry type of the stiffness tensor!

Observation

The usual pullback of a tensor field does not work: typically $\phi^* F_a^{qP} \neq F_{\phi^* a}^{qP}$.
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Changing the physical model (symmetry type) fundamentally changes the result.

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