JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

Imaging the universe with a gaugeless theory of conformal geometry

MATH + X Symposium, Hella, Iceland

Joonas Ilmavirta

June 1, 2023

Based on joint work with
Maarten de Hoop, Henri Hänninen, Matti Lassas, Teemu Saksala

Prelude

Question

Can a color-blind astronomer get a reliable model of the universe by just looking at the sky?

Prelude

Question

Can a color-blind astronomer get a reliable model of the universe by just looking at the sky?

Question

If one makes geometric measurements of the arrivals of all photons from all supernova explosions but has no spectral information on the photons, can one reconstruct the Lorentzian metric describing the spacetime in one's visible past?

Prelude

Question

Can a color-blind astronomer get a reliable model of the universe by just looking at the sky?

Question

If one makes geometric measurements of the arrivals of all photons from all supernova explosions but has no spectral information on the photons, can one reconstruct the Lorentzian metric describing the spacetime in one's visible past?

No!

Prelude

Question

Can a color-blind astronomer get a reliable model of the universe by just looking at the sky?

Question

If one makes geometric measurements of the arrivals of all photons from all supernova explosions but has no spectral information on the photons, can one reconstruct the Lorentzian metric describing the spacetime in one's visible past?

No! With conformally invariant data one has no sense of scale.

Prelude

Question

Can a color-blind astronomer get a reliable model of the universe by just looking at the sky?

Question

If one makes geometric measurements of the arrivals of all photons from all supernova explosions but has no spectral information on the photons, can one reconstruct the Lorentzian metric describing the spacetime in one's visible past?

No! With conformally invariant data one has no sense of scale.

Question

What geometric structures best describe the setup?

Outline

1. General relativity
   - Lorentz manifolds
   - Equations of motion
   - Conformal symmetry
   - Comparison to Riemannian geometry
2. Inverse problems
3. Free geometry

Lorentz manifolds

- Euclidean space: $\mathbb{R}^n$ with the quadratic form $x_1^2 + x_2^2 + \cdots + x_n^2 =: |x|^2$

Lorentz manifolds

- Euclidean space: $\mathbb{R}^n$ with the quadratic form $x_1^2 + x_2^2 + \cdots + x_n^2 =: |x|^2$
- Minkowski space: $\mathbb{R}^n$ with the quadratic form $x_1^2 - x_2^2 - \cdots - x_n^2 =: |x|^2$

Lorentz manifolds

- Euclidean space: $\mathbb{R}^n$ with the quadratic form $x_1^2 + x_2^2 + \cdots + x_n^2 =: |x|^2$
- Minkowski space: $\mathbb{R}^n$ with the quadratic form $x_1^2 - x_2^2 - \cdots - x_n^2 =: |x|^2$
- Riemannian manifold: Smooth manifold where the infinitesimal local geometry is Euclidean.

Lorentz manifolds

- Euclidean space: $\mathbb{R}^n$ with the quadratic form $x_1^2 + x_2^2 + \cdots + x_n^2 =: |x|^2$
- Minkowski space: $\mathbb{R}^n$ with the quadratic form $x_1^2 - x_2^2 - \cdots - x_n^2 =: |x|^2$
- Riemannian manifold: Smooth manifold where the infinitesimal local geometry is Euclidean.
- Lorentzian manifold: Smooth manifold where the infinitesimal local geometry is Minkowskian.

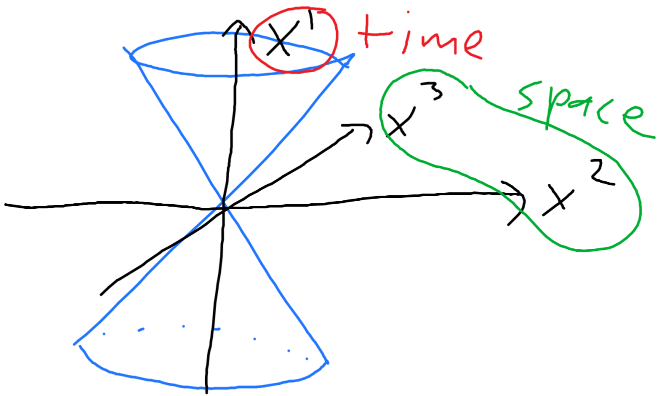
Lorentz manifolds

- Euclidean space: $\mathbb{R}^n$ with the quadratic form $x_1^2 + x_2^2 + \cdots + x_n^2 =: |x|^2$
- Minkowski space: $\mathbb{R}^n$ with the quadratic form $x_1^2 - x_2^2 - \cdots - x_n^2 =: |x|^2$
- Riemannian manifold: Smooth manifold where the infinitesimal local geometry is Euclidean.
- Lorentzian manifold: Smooth manifold where the infinitesimal local geometry is Minkowskian.
- Special relativity lives on a Minkowski space.

Lorentz manifolds

- Euclidean space: $\mathbb{R}^n$ with the quadratic form $x_1^2 + x_2^2 + \cdots + x_n^2 =: |x|^2$
- Minkowski space: $\mathbb{R}^n$ with the quadratic form $x_1^2 - x_2^2 - \cdots - x_n^2 =: |x|^2$
- Riemannian manifold: Smooth manifold where the infinitesimal local geometry is Euclidean.
- Lorentzian manifold: Smooth manifold where the infinitesimal local geometry is Minkowskian.
- Special relativity lives on a Minkowski space. General relativity lives on a Lorentzian manifold.

Lorentz manifolds



The light cone $L = \{x \in \mathbb{R}^n; |x|^2 = 0\}$
is the set of all possible 4-velocities of a photon.

Equations of motion

There are two equations of motion:

Equations of motion

There are two equations of motion:

1. Geodesic equation for matter motion
2. Einstein's field equation for geometry

Equations of motion

There are two equations of motion:

1. Geodesic equation for matter motion
2. Einstein's field equation for geometry

I will be interested in the kinematics of photons (and neutrinos) and the relevant geometry, not the interpretation of that geometry.

Equations of motion

There are two equations of motion:

1. Geodesic equation for matter motion
2. Einstein's field equation for geometry

I will be interested in the kinematics of photons (and neutrinos) and the relevant geometry, not the interpretation of that geometry.

What matters is that particles follow geodesics, and photons follow lightlike geodesics: $|\dot{\gamma}|^2 = 0$.

Conformal symmetry

- Two Lorentzian metrics g and h are conformal ($g \sim h$) if $g = ch$ for a scalar $c > 0$.

Conformal symmetry

- Two Lorentzian metrics g and h are conformal ($g \sim h$) if $g = ch$ for a scalar $c > 0$.
- The conformal class of g is $[g] = \{h; h \sim g\}$.

Conformal symmetry

- Two Lorentzian metrics g and h are conformal ($g \sim h$) if $g = ch$ for a scalar $c > 0$.
- The conformal class of g is $[g] = \{h; h \sim g\}$.
- Invariants of a conformal class:

Conformal symmetry

- Two Lorentzian metrics g and h are conformal ($g \sim h$) if $g = ch$ for a scalar $c > 0$.
- The conformal class of g is $[g] = \{h; h \sim g\}$.
- Invariants of a conformal class:
  - Light cone bundle.

Conformal symmetry

- Two Lorentzian metrics g and h are conformal ($g \sim h$) if $g = ch$ for a scalar $c > 0$.
- The conformal class of g is $[g] = \{h; h \sim g\}$.
- Invariants of a conformal class:
  - Light cone bundle. — Equivalent information!

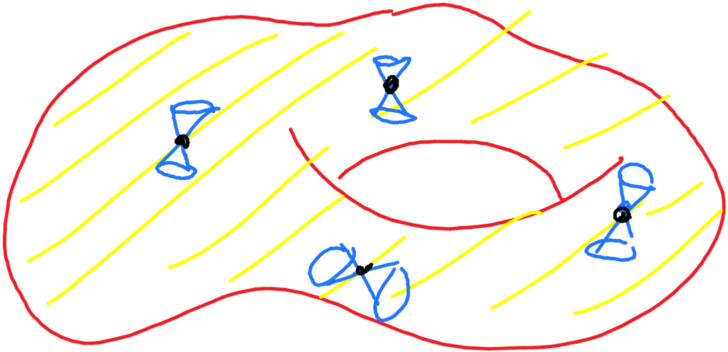
Conformal symmetry

- Two Lorentzian metrics g and h are conformal ($g \sim h$) if $g = ch$ for a scalar $c > 0$.
- The conformal class of g is $[g] = \{h; h \sim g\}$.
- Invariants of a conformal class:
  - Light cone bundle. — Equivalent information!
  - Lightlike geodesics as sets.

Conformal symmetry

- Two Lorentzian metrics g and h are conformal ($g \sim h$) if $g = ch$ for a scalar $c > 0$.
- The conformal class of g is $[g] = \{h; h \sim g\}$.
- Invariants of a conformal class:
  - Light cone bundle. — Equivalent information!
  - Lightlike geodesics as sets. — Parametrization is irrelevant!

Conformal symmetry



The light cone bundle of a spacetime is the collection of light cones at all points.
$L_xM = \{v \in T_xM; |v|^2 = 0\}$ and $LM = \{v \in TM; |v|^2 = 0\}$.

Comparison to Riemannian geometry

Lorentzian geometry:

Riemannian geometry:

Comparison to Riemannian geometry

Lorentzian geometry:

- At any $x \in M$ on T_xM:

Riemannian geometry:

- At any $x \in M$ on T_xM:

Comparison to Riemannian geometry

Lorentzian geometry:

- At any $x \in M$ on T_xM:
  - Quadratic form that looks like $v \mapsto v_1^2 - v_2^2 - \cdots - v_n^2 =: |v|^2$.

Riemannian geometry:

- At any $x \in M$ on T_xM:
  - Quadratic form that looks like $v \mapsto v_1^2 + v_2^2 + \cdots + v_n^2 =: |v|^2$.

Comparison to Riemannian geometry

Lorentzian geometry:

- At any $x \in M$ on T_xM:
  - Quadratic form that looks like $v \mapsto v_1^2 - v_2^2 - \cdots - v_n^2 =: |v|^2$.
  - Light cone $L_xM = \{v \in T_xM; |v|^2 = 0\}$.

Riemannian geometry:

- At any $x \in M$ on T_xM:
  - Quadratic form that looks like $v \mapsto v_1^2 + v_2^2 + \cdots + v_n^2 =: |v|^2$.
  - Unit sphere $S_xM = \{v \in T_xM; |v|^2 = 1\}$.

Comparison to Riemannian geometry

Lorentzian geometry:

- At any $x \in M$ on T_xM:
  - Quadratic form that looks like $v \mapsto v_1^2 - v_2^2 - \cdots - v_n^2 =: |v|^2$.
  - Light cone $L_xM = \{v \in T_xM; |v|^2 = 0\}$.
- Light cone bundle $LM = \bigcup_{x \in M} L_xM$.

Riemannian geometry:

- At any $x \in M$ on T_xM:
  - Quadratic form that looks like $v \mapsto v_1^2 + v_2^2 + \cdots + v_n^2 =: |v|^2$.
  - Unit sphere $S_xM = \{v \in T_xM; |v|^2 = 1\}$.
- Unit sphere bundle $SM = \bigcup_{x \in M} S_xM$.

Comparison to Riemannian geometry

Lorentzian geometry:

- At any $x \in M$ on T_xM:
  - Quadratic form that looks like $v \mapsto v_1^2 - v_2^2 - \cdots - v_n^2 =: |v|^2$.
  - Light cone $L_xM = \{v \in T_xM; |v|^2 = 0\}$.
- Light cone bundle $LM = \bigcup_{x \in M} L_xM$.
- $LM \iff$ the conformal class of g.

Riemannian geometry:

- At any $x \in M$ on T_xM:
  - Quadratic form that looks like $v \mapsto v_1^2 + v_2^2 + \cdots + v_n^2 =: |v|^2$.
  - Unit sphere $S_xM = \{v \in T_xM; |v|^2 = 1\}$.
- Unit sphere bundle $SM = \bigcup_{x \in M} S_xM$.
- $SM \iff g$.

Comparison to Riemannian geometry

Lorentzian geometry:

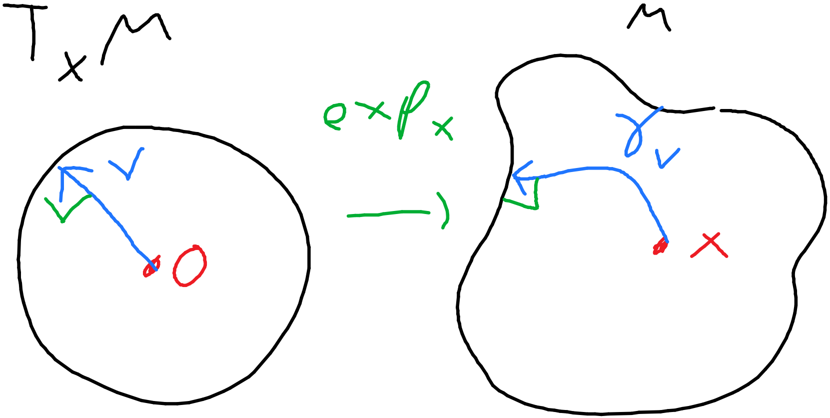
- At any $x \in M$ on T_xM:
  - Quadratic form that looks like $v \mapsto v_1^2 - v_2^2 - \cdots - v_n^2 =: |v|^2$.
  - Light cone $L_xM = \{v \in T_xM; |v|^2 = 0\}$.
- Light cone bundle $LM = \bigcup_{x\in M} L_xM$.
- $LM \iff$ the conformal class of g.

Riemannian geometry:

- At any $x \in M$ on T_xM:
  - Quadratic form that looks like $v \mapsto v_1^2 + v_2^2 + \cdots + v_n^2 =: |v|^2$.
  - Unit sphere $S_xM = \{v \in T_xM; |v|^2 = 1\}$.
- Unit sphere bundle $SM = \bigcup_{x\in M} S_xM$.
- $SM \iff g$.

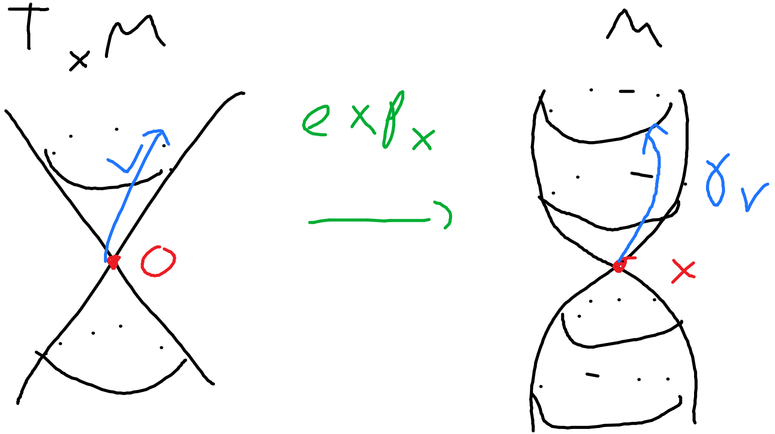
If (M, g) is a Riemannian manifold and $(\mathbb{R} \times M, \mathrm{d}t^2 - g)$ is a Lorentzian manifold, then Riemannian and lightlike geodesics match.

Comparison to Riemannian geometry



Riemannian Gauss's lemma:
Geodesics are the normal direction to spheres.

Comparison to Riemannian geometry



Lorentzian Gauss's lemma:

Lightlike geodesics are the normal (and tangent!) direction to light cones.

Outline

Measurements

- There are active and passive measurements.

Measurements

- There are active and passive measurements.
  Cosmological scales require passivity.

Measurements

- There are active and passive measurements. Cosmological scales require passivity.
- We need sources in spacetime, not in space.

Measurements

- There are active and passive measurements. Cosmological scales require passivity.
- We need sources in spacetime, not in space.
- The sources should be dense.

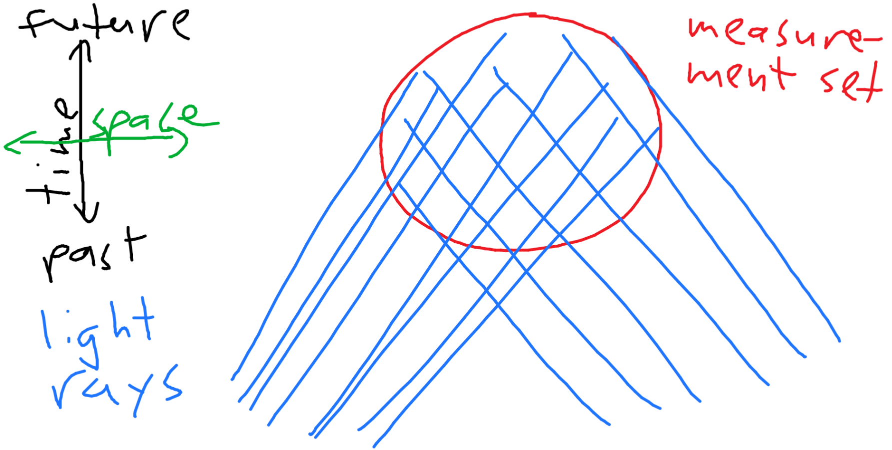
Measurements

- There are active and passive measurements.
  Cosmological scales require passivity.
- We need sources in spacetime, not in space.
- The sources should be dense.
  The density of supernovae is roughly $\frac{1\text{ supernova}}{(10^5\text{ years})^4}$.

Measurements

- There are active and passive measurements.
  Cosmological scales require passivity.
- We need sources in spacetime, not in space.
- The sources should be dense.
  The density of supernovae is roughly $\frac{1 \text{ supernova}}{(10^5 \text{ years})^4}$.
- The color-blind measurement consists of the observed light cones in an open measurement set $U \subset M$.

Measurements



The visible past of a measurement set $U \subset M$.

Results

Results

Theorem (Kurylev–Lassas–Uhlmann, 2018)

Measurements of light cones in an open subset of the spacetime determine the geometry and conformal class of the spacetime in the visible past of the measurement set.

Results

Theorem (Kurylev–Lassas–Uhlmann, 2018)

Measurements of light cones in an open subset of the spacetime determine the geometry and conformal class of the spacetime in the visible past of the measurement set.

This is optimal!

Results

Theorem (Kurylev–Lassas–Uhlmann, 2018)

Measurements of light cones in an open subset of the spacetime determine the geometry and conformal class of the spacetime in the visible past of the measurement set.

This is optimal!

Theorem (I.–Uhlmann, 2021)

Suppose the conformal class is known. Measurements of (perturbative) neutrino cones in an open subset of the spacetime determine the conformal factor in the visible past of the measurement set.

Results

Theorem (Kurylev–Lassas–Uhlmann, 2018)

Measurements of light cones in an open subset of the spacetime determine the geometry and conformal class of the spacetime in the visible past of the measurement set.

This is optimal!

Theorem (I.–Uhlmann, 2021)

Suppose the conformal class is known. Measurements of (perturbative) neutrino cones in an open subset of the spacetime determine the conformal factor in the visible past of the measurement set.

Photons and neutrinos together determine the full geometry of the visible part of the spacetime!

Results

The light and neutrino cones of a supernova.

Future directions

- A hurdle in modelling: Realistic measurements are local and passive.

Future directions

- A hurdle in modelling: Realistic measurements are local and passive.
- The area of cosmological inverse problems is underdeveloped.

Future directions

- A hurdle in modelling: Realistic measurements are local and passive.
- The area of cosmological inverse problems is underdeveloped.
- The useful tools are underdeveloped.

Future directions

- A hurdle in modelling: Realistic measurements are local and passive.
- The area of cosmological inverse problems is underdeveloped.
- The useful tools are underdeveloped.
- Mathematical open problems, for both formulation and solution:

Future directions

- A hurdle in modelling: Realistic measurements are local and passive.
- The area of cosmological inverse problems is underdeveloped.
- The useful tools are underdeveloped.
- Mathematical open problems, for both formulation and solution: pretty much all of cosmology.

Future directions

- A hurdle in modelling: Realistic measurements are local and passive.
- The area of cosmological inverse problems is underdeveloped.
- The useful tools are underdeveloped.
- Mathematical open problems, for both formulation and solution:
  pretty much all of cosmology.
- The leading order model will probably often be conformally invariant.

Outline

Gaugelessness

- Gauge symmetry: The gauge transformation $g \mapsto cg$ leaves lightlike geodesics intact.

Gaugelessness

- Gauge symmetry: The gauge transformation $g \mapsto cg$ leaves lightlike geodesics intact.
- Typical formulation in $[g]$: Pick an example metric $h \in [g]$ and study its geodesics.
  I.e., fix the gauge.

Gaugelessness

- Gauge symmetry: The gauge transformation $g \mapsto cg$ leaves lightlike geodesics intact.
- Typical formulation in $[g]$: Pick an example metric $h \in [g]$ and study its geodesics. I.e., fix the gauge.
- We want everything to be uniquely defined without choosing a gauge.

Gaugelessness

- Gauge symmetry: The gauge transformation $g \mapsto cg$ leaves lightlike geodesics intact.
- Typical formulation in $[g]$: Pick an example metric $h \in [g]$ and study its geodesics. I.e., fix the gauge.
- We want everything to be uniquely defined without choosing a gauge.
- We want all of the theory to be conformally invariant: every object, every equation, every theorem. . .

Gaugelessness

- Gauge symmetry: The gauge transformation $g \mapsto cg$ leaves lightlike geodesics intact.
- Typical formulation in $[g]$: Pick an example metric $h \in [g]$ and study its geodesics. I.e., fix the gauge.
- We want everything to be uniquely defined without choosing a gauge.
- We want all of the theory to be conformally invariant: every object, every equation, every theorem…
- Such a conformal geometry is free of gauge.

The free geodesic equation

- The usual lightlike geodesic equation: $D_t^2\gamma(t) = 0$.

The free geodesic equation

- The usual lightlike geodesic equation: $D_t^2\gamma(t) = 0$.
- The free lightlike geodesic equation: $D_t^2\gamma(t) \in \mathbb{R}\dot\gamma(t)$.

The free geodesic equation

- The usual lightlike geodesic equation: $D_t^2\gamma(t) = 0$.
- The free lightlike geodesic equation: $D_t^2\gamma(t) \in \mathbb{R}\dot{\gamma}(t)$.
- Key properties: Parametrization and conformal class do not matter.

The free geodesic equation

- The usual lightlike geodesic equation: $D_t^2\gamma(t) = 0$.
- The free lightlike geodesic equation: $D_t^2\gamma(t) \in \mathbb{R}\dot{\gamma}(t)$.
- Key properties: Parametrization and conformal class do not matter.
- Timelike and spacelike geodesics do not behave well.

The free geodesic equation

- The usual lightlike geodesic equation: $D_t^2\gamma(t) = 0$.
- The free lightlike geodesic equation: $D_t^2\gamma(t) \in \mathbb{R}\dot{\gamma}(t)$.
- Key properties: Parametrization and conformal class do not matter.
- Timelike and spacelike geodesics do not behave well.
- Invariant bundle with connection: Pull back (and rescale) TM over γ, take the subbundle $\dot{\gamma}^\perp$ and quotient by $\mathbb{R}\dot{\gamma}$.

The free geodesic equation

- The usual lightlike geodesic equation: $D_t^2\gamma(t) = 0$.
- The free lightlike geodesic equation: $D_t^2\gamma(t) \in \mathbb{R}\dot{\gamma}(t)$.
- Key properties: Parametrization and conformal class do not matter.
- Timelike and spacelike geodesics do not behave well.
- Invariant bundle with connection: Pull back (and rescale) TM over γ, take the subbundle $\dot{\gamma}^\perp$ and quotient by $\mathbb{R}\dot{\gamma}$.
- . . . and similarly for other definitions — when they work.

Singularities

- Famous singularities, black holes and the big bang are different:

Singularities

- Famous singularities, black holes and the big bang are different:
  - The big bang is conformally smooth: $g = ch$, where h is smooth and c is singular.
  - A black hole is fully singular: no conformal rescaling is smooth.

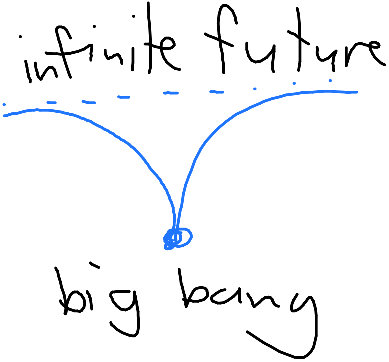
Singularities

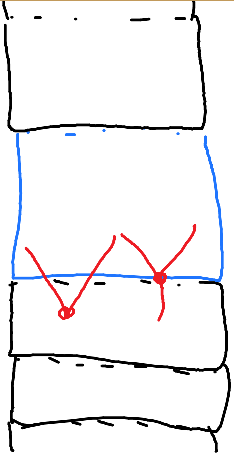
- Famous singularities, black holes and the big bang are different:
  - The big bang is conformally smooth: $g = ch$, where h is smooth and c is singular.
  - A black hole is fully singular: no conformal rescaling is smooth.
- For free geometry the big bang is smooth but black holes are singular.

Singularities

- Famous singularities, black holes and the big bang are different:
  - The big bang is conformally smooth: $g = ch$, where h is smooth and c is singular.
  - A black hole is fully singular: no conformal rescaling is smooth.
- For free geometry the big bang is smooth but black holes are singular.
- If massive fields (or the vacuum) decay in the far future, then both our asymptotic past and asymptotic future are mostly scale-free.

Singularities





Penrose's conformal cyclic cosmology with aeons separated by inflation.

Our aeon could see signals from the previous aeon, such as Hawking points.

Invitation

Invitation

I would welcome more interaction between cosmology and geometric inverse problems:

Invitation

I would welcome more interaction between cosmology and geometric inverse problems:

- Formulate cosmological questions mathematically.

Invitation

I would welcome more interaction between cosmology and geometric inverse problems:

- Formulate cosmological questions mathematically.
- Find a balance between realism and mathematical tractability.

Invitation

I would welcome more interaction between cosmology and geometric inverse problems:

- Formulate cosmological questions mathematically.
- Find a balance between realism and mathematical tractability.
- Find useful structures that allow proving theorems.

Invitation

I would welcome more interaction between cosmology and geometric inverse problems:

- Formulate cosmological questions mathematically.
- Find a balance between realism and mathematical tractability.
- Find useful structures that allow proving theorems.
- And maybe prove some theorems.

Recap

- Conformal equivalence: $g = ch$ with a scalar c.
- Conformal class $=$ equivalence class.
- Color-blind measurements of photons are conformally invariant.
- There are theorems about unique recovery of the conformal class. . .
- . . . but very few.
- There is a gaugeless theory of conformal geometry that allows looking back at the universe along lightlike geodesics.
- Stay tuned. . .

DISCOVERING MATH at JYU. Since 1863.

Slides and papers available:

`http://users.jyu.fi/~jojapeil`

Ask for details:

`joonas.ilmavirta@jyu.fi`