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Two-layer tomography assisted by anisotropy

Geo-Mathematical Imaging Group Project Review

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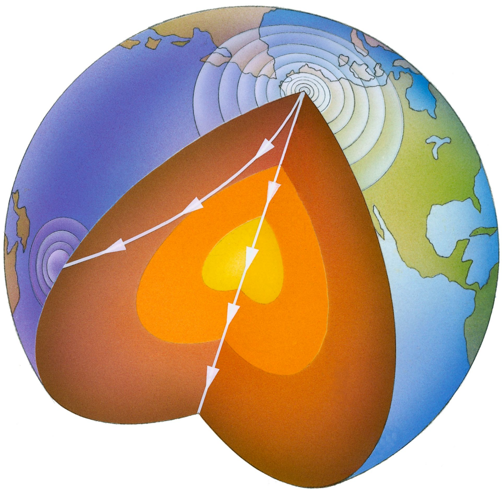
Based on joint work with
Maarten de Hoop, Matti Lassas, Anthony Várilly-Alvarado

The question



How to see the interior of the Earth via seismic rays?

Today's highlights

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Theorem (de Hoop–I–Lassas–Várilly-Alvarado 2023)

Suppose the planet is piecewise homogeneous (but anisotropic) with two layers. Measurements of travel times of qP rays generically determine the whole model:

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Theorem (de Hoop–I–Lassas–Várilly-Alvarado 2023)

Generically an anisotropic stiffness tensor is uniquely determined by any of the following:

- *slowness polynomial,*
- *slowness surface,*
- *a small part of the qP branch of the slowness surface.*

But orthorhombic stiffness tensors are not unique!

Outline

Elastic wave equation

Quantities:

- Displacement $u(t, x) \in \mathbb{R}^n$.
- Density $\rho(x) \in \mathbb{R}$.
- Stiffness tensor $c_{ijkl}(x) \in \mathbb{R}^{n^4}$.

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Properties:

- $\rho > 0$.
- $c_{ijkl} = c_{klij} = c_{jikl}$.
- $\sum_{i,j,k,l} c_{ijkl} A_{ij} A_{kl} > 0$ whenever $A = A^T \neq 0$.

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Equation of motion:
$$\rho(x)\partial_t^2 u_i(t,x) - \sum_{j,k,l} \partial_j [c_{ijkl}(x)\partial_k u_l(x)] = 0.$$

Propagation of singularities

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To understand singularities of solutions to the EWE, freeze ρ and c to be constants.
If $u = Ae^{i\omega(t-p\cdot x)}$, then the EWE becomes

$$\rho\omega^2[-I + \Gamma(p)]A = 0,$$

where

$$\Gamma_{il}(p) = \sum_{j,k} \rho^{-1} c_{ijkl} p_j p_k$$

is the Christoffel matrix.

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The singularities move according to the geodesic flow of the Finsler geometry given by $F^{qP} = [\lambda_1(\Gamma)^{1/2}]^*$.

Slowness polynomial and slowness surface

A reduced stiffness tensor $a_{ijkl} = \rho^{-1}_{ijkl}$ defines

- a Christoffel matrix $\Gamma_a(p)$ and
- a slowness polynomial $P_a(p) = \det[\Gamma_a(p) - I]$.

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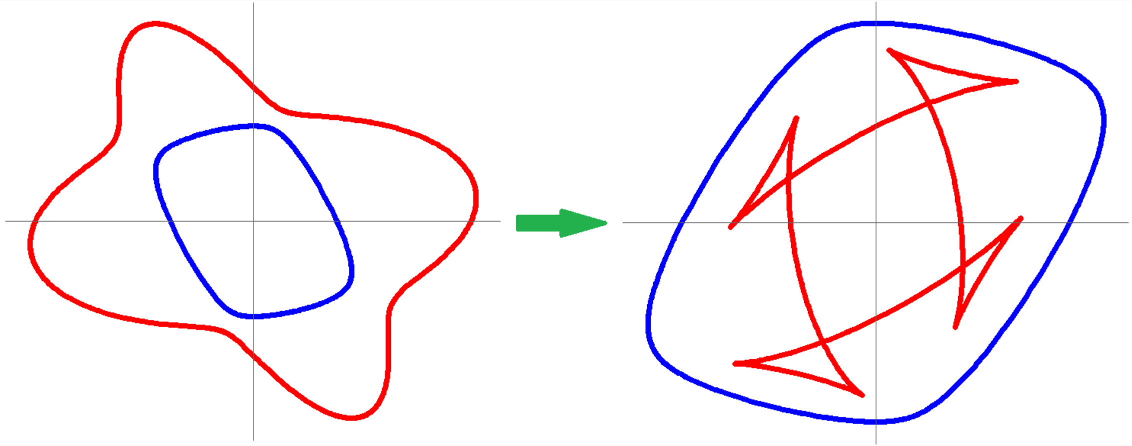
The set where singularities are possible is the slowness surface

$$\Sigma_a = \{p \in \mathbb{R}^n; P_a(p) = 0\}.$$

Knowing the slowness polynomial $\iff$ knowing the slowness surface.

Slowness polynomial and slowness surface



A slowness surface in 2D with its two branches, and the corresponding two Finsler norms.

The quasi pressure (qP) polarization behaves well.

Anisotropy $\iff$ dependence on direction $\iff$ not circles.

Geometrization of an analytic problem

Original inverse problem

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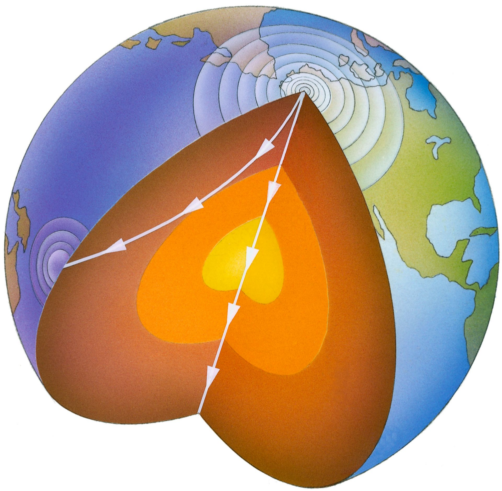
Remarks:

- Geometric inverse problems like this can be solved for qP geometries.
- Riemannian geometry is not enough; it can only handle a tiny subclass of physically valid and interesting stiffness tensors.
- Knowing the metric is the same as knowing the (co)sphere bundle: (M, g) or $(M, F) \iff (M, SM) \iff (M, S^*M)$.
- The cospheres of the Finsler geometry are the qP branches of the slowness surface.

Geometrization of an analytic problem



Rays follow geodesics and tell about the interior structure encoded as a geometry.

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Algebraic variety

Definition

A set $V \subset \mathbb{R}^n$ is an algebraic variety if it is the vanishing set of a collection of polynomials $\mathbb{R}^n \to \mathbb{R}$.

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The slowness surface is the vanishing set of the slowness polynomial and thus a variety.

The study of the geometry of varieties is a part of algebraic geometry.

Generic irreducibility

Definition

A variety $V \subset \mathbb{R}^n$ is reducible if it can be written as the union of two varieties in a non-trivial way.

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This is not true for all a.

Generic irreducibility

Corollary (de Hoop, Ilmavirta, Lassas, Várilly-Alvarado)

When the slowness surface Σ_a is irreducible, any (Euclidean) relatively open subset determines the whole slowness surface.
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It suffices to measure the well-behaved qP branch!

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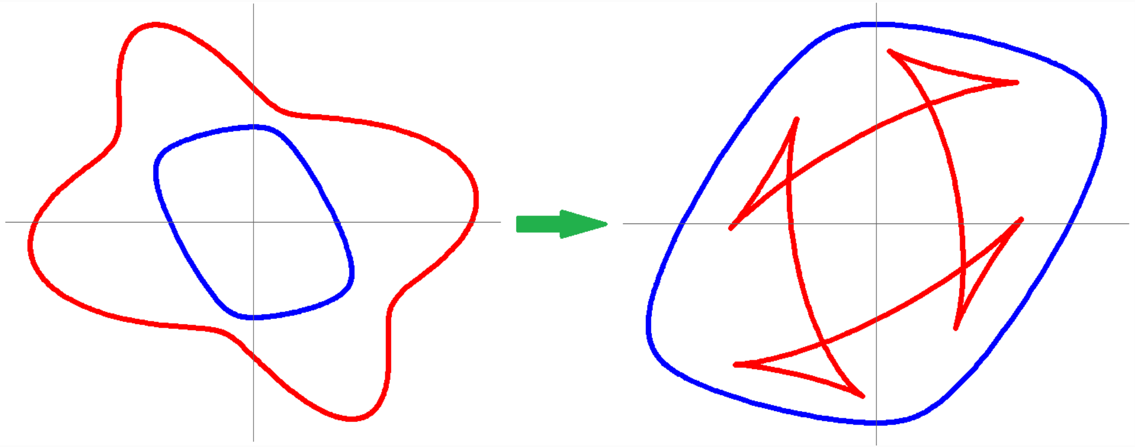

Any small part of the well-behaved quasi pressure branch
determines the whole thing via Zariski closure.

Generically unique reduced stiffness tensor

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Corollary (de Hoop–Ilmavirta–Lassas–Várilly-Alvarado)

Let $n \in \{2, 3\}$. Generically any small subset of the slowness surface Σ_a determines a.

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Measurement: Travel times and directions of waves between all surface points, for all polarizations.

Result: The measurement generically determines the model completely!

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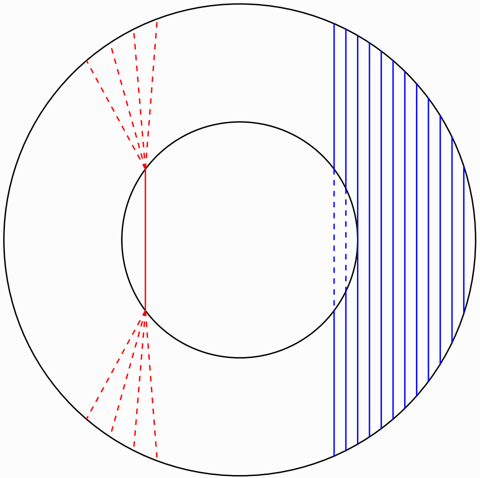
- Take rays deeper and deeper and see when they start behaving oddly.
- Get the interface between the layers.
- Aim rays at any two points on the interface and get their distance in travel time.
- Get a bit of the qP group velocity surface in the core.
- Repeat the above to get the stiffness tensor in the core — generically.

The proof



First find outer stiffness and boundary, then inner stiffness.

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