JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

Geometry and unexpected coupling of slowness surfaces

Geo-Mathematical Imaging Group project review meeting

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Based on joint work with
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Finnish Centre of Excellence in Inverse Modelling and Imaging

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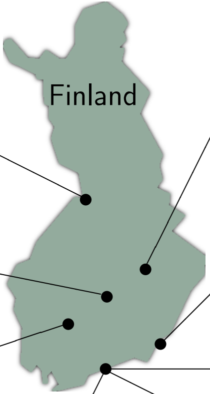
CENTRES OF EXCELLENCE IN RESEARCH



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Topics and goals

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- Elastic geometry and slowness surfaces.

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- Algebraic geometry.

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- Elastic geometry and slowness surfaces.
- Algebraic geometry.
- These two connect to help solve inverse problems.

Outline

Newton's gravitation

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- The gravitational force exerted by the Sun causes the Earth's trajectory to curve.
- The force is described by a simple formula and the equation of motion is an ODE in $\mathbb{R}^3$.
- The Newtonian approach is straightforward to use and often a good model.

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- There is a relatively simple equation of motion for the planet: The geodesic equation is a non-linear ODE.
- There is a complicated equation of motion for the geometry itself: Einstein's field equation is a non-linear system of coupled PDEs.
- This model is harder to use but can reach phenomena inaccessible to Newtonian gravity and provides a more geometric way to see the essential structures.

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- The particles of the elastic displacement field are called *phonons*.
  - Traditional view: The trajectory of the phonon is curved because wave speed varies.
  - Newer view: The phonon goes straight in a curved geometry (along a geodesic), and the geometry is curved by variations in wave speed.

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     (The slowness surface is convex.)
- A Riemannian manifold has an inner product at every point.
  A Finsler manifold has a norm at every point.

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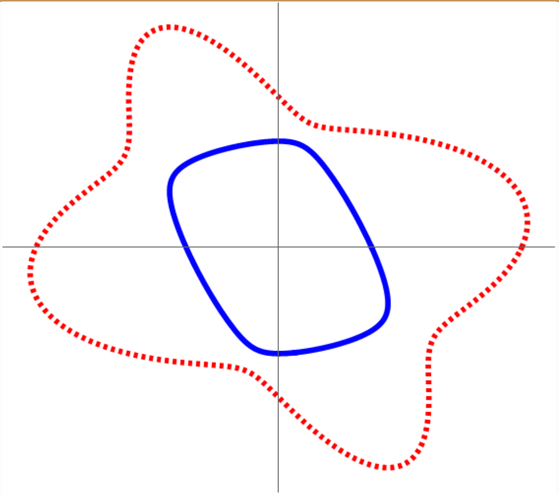
- Christoffel matrix: $\Gamma_{jk}(p) = \sum_{il} a_{ijkl} p_i p_k$.
- Slowness polynomial: $P_a(p) = \det(\Gamma(p) - I)$.
- Slowness surface: Those $p \in \mathbb{R}^n$ for which $P_a(p) = 0$.

The slowness surface



Slowness surface in 2D. Pressure and shear branches.
Only qP is convex.

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How about the stiffness tensor?

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Not for all stiffness tensors but yes for most of them.

The set of stiffness tensors for which this works is generic: it contains an open and dense subset.

Zariski topology

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- The Zariski topology is incompatible with differential geometry and analysis, and it is not Hausdorff, but it is perfect for describing the geometry of zero sets of polynomials.
- The slowness surface is a zero set of a polynomial!

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- Cheap: The Euclidean closure $\bar{\sigma}$ is also in Σ.
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- Definition of Zariski closure: If $f|_\sigma = 0 \implies f(x) = 0$, then $x \in \bar{\sigma}$.

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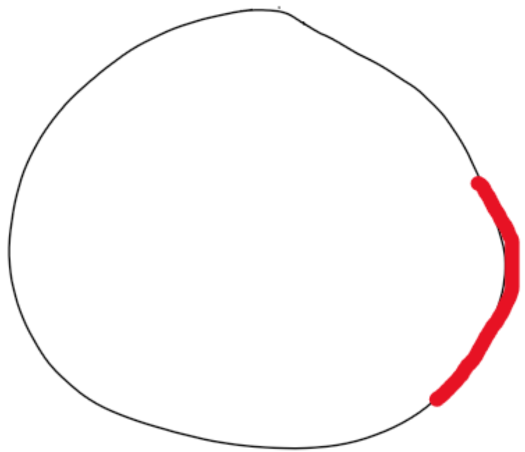
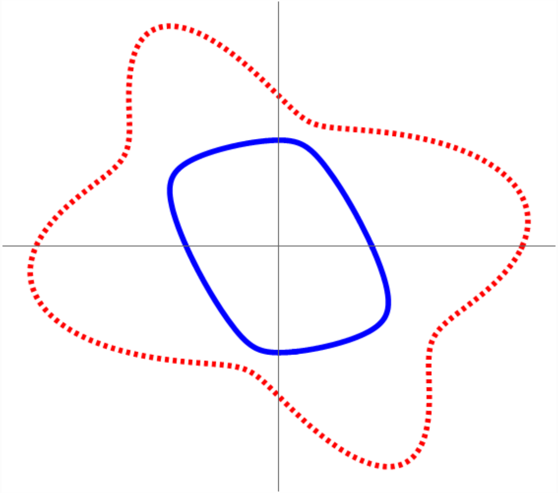


The Zariski closure of a circular arc is the whole circle.

Zariski topology



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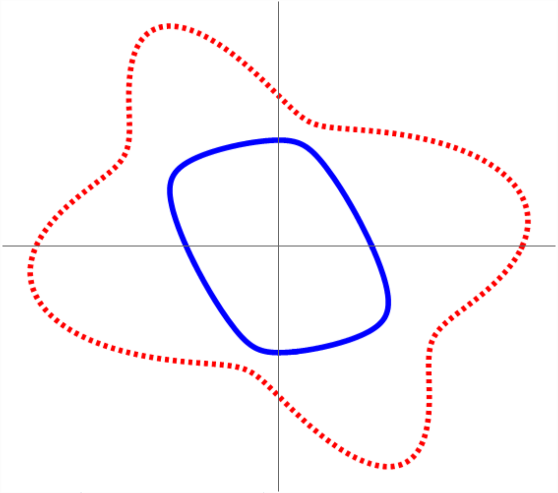

The Zariski closure of a small part of the qP branch is the whole qP branch.
If P_a is irreducible, the closure is the whole slowness surface!

Technical result

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Let $n = 2$ or $n = 3$. The set of those stiffness tensors a for which the slowness polynomial is irreducible contains a non-empty Zariski-open subset.

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The previous theorem follows.

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- Both inversion steps are efficiently and easily implemented but hard to prove.
- Most polynomials are irreducible, but slowness polynomials are a very special class of polynomials with some odd properties.
  We don't fully understand all the special structure slowness surfaces have as varieties.
- We use relatively heavy tools in algebraic geometry.

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The tensor is not determined uniquely by the data if it is isotropic!

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Suppose the stiffness tensor inside a domain is anisotropic and piecewise homogeneous with a convex interface.
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If we measure fastest arrival times on the surface, we generically recover both stiffness tensors and the interface.

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Stay tuned for the next GMIG workshop!

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