Geometric inverse problems arising from geophysics

UCI inverse problems seminar

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6 May 2021

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physics

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PDE

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geometry

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inverse problems

$\downarrow$

physics

Outline

Newton's theory

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- The gravitational force exerted by the Sun causes the Earth's trajectory to curve.
- The force is described by a simple formula and the equation of motion is an ODE in $\mathbb{R}^3$.
- The Newtonian approach is straightforward to use and often a good model.

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- There is a relatively simple equation of motion for the planet: The geodesic equation is a non-linear ODE.
- There is a complicated equation of motion for the geometry itself: Einstein's field equation is a non-linear system of coupled PDEs.
- This model is harder to use but can reach phenomena inaccessible to Newtonian gravity and provides a more geometric way to see the essential structures.

The goal

A geometric theory of elasticity?

Untoy the model: Bring mathematics closer to the application.

Outline

1. Geometrization of gravitation
2. Elastic waves
   - The stiffness tensor
   - The elastic wave equation
   - The principal symbol
   - Polarization
   - Singularities and the slowness surface
3. Elastic geometry
4. Inverse problems

The stiffness tensor

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- Density normalized: $a_{ijkl}(x) = c_{ijkl}(x)/\rho(x)$.

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- If the material is anisotropic (c is no more symmetric than necessary and wave speed depends on direction), then the vector nature of the equation cannot be ignored.
- Elastic waves arising from earthquakes (or marsquakes!) satisfy this equation away from the focus of the event to great accuracy. (Weak field limit.)

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- The principal symbol of the EWO is $\Gamma(x,\xi) - \omega^2 I$, where $\xi = \omega p$.

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- Polarization vectors are eigenvectors of the Christoffel matrix Γ, so they are orthogonal.

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- The admissible slowness vectors p are on the slowness surface given by the equation

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- There are two geometries: “spatial” and “temporal”.

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- Compare to gravitation!

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- In elastic geometry we measure distance in travel time, and the waves go straight in this geometry.
- Fermat’s principle: Phonons — the particles corresponding to elastic waves — go straight in the geometry given by travel time.
- Fermat’s principle is about going straight in the relevant geometry, not about taking the shortest path. These are not the same thing over long distances or for shear waves.

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- The norms on the dual spaces T_x^*M satisfy the same conditions.

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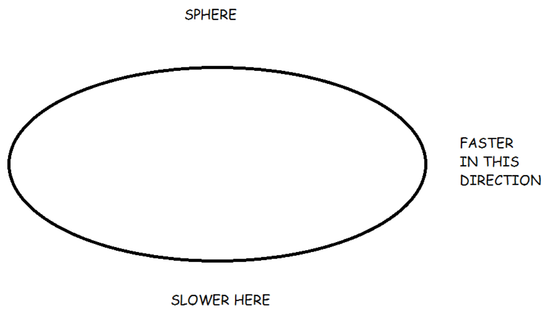
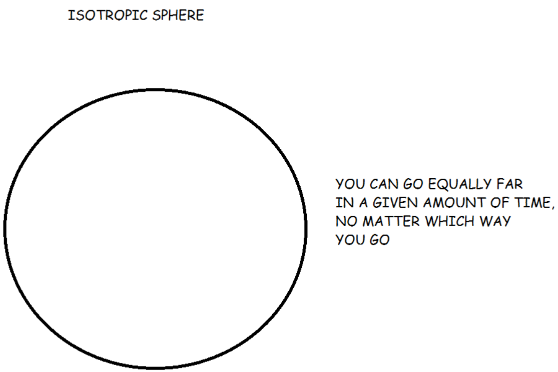
Other polarizations are problematic.

Spheres and cospheres



Sphere of possible group velocities.

Spheres and cospheres



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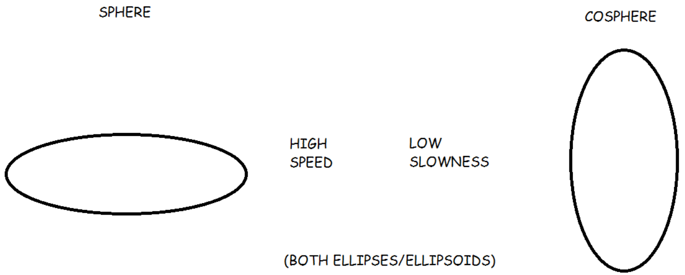
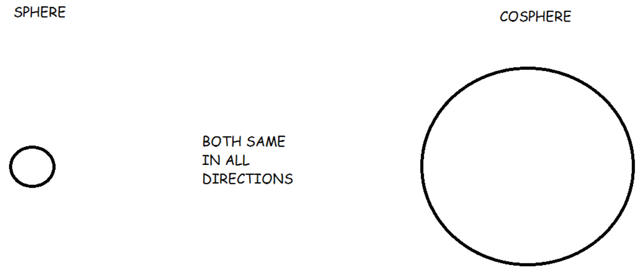


Sphere of possible group velocities and cosphere of possible phase velocities.
Elliptic or Riemannian.

Spheres and cospheres



Sphere of possible group velocities and cosphere of possible phase velocities.
Isotropic.

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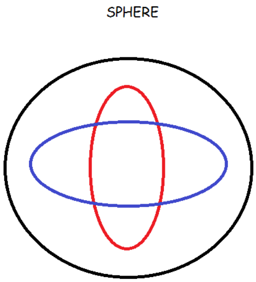

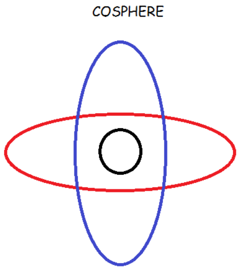

Three polarizations with their own group and phase velocities.
Black is qP, red and blue are qS.

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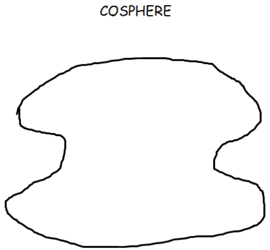

Non-convex cosphere.

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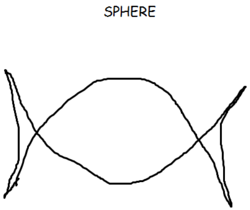

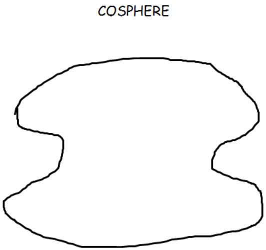

Non-convex duality. Triplication.

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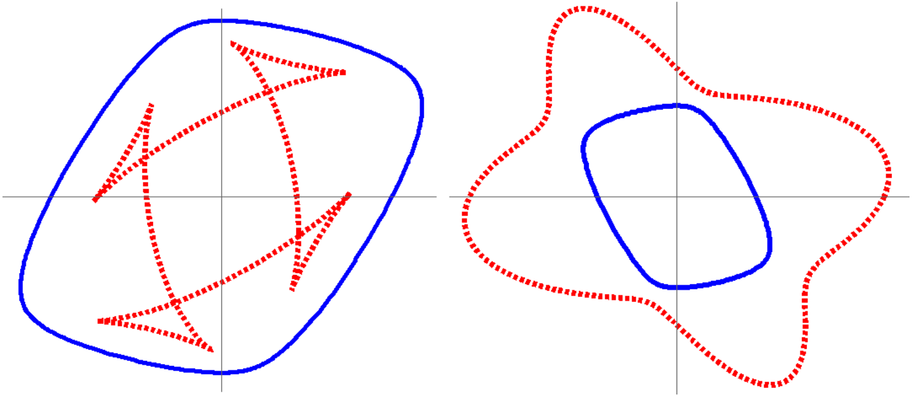

Realistic example in 2D.

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Modelling goal

Building a complete theory of elastic geometry — a direct link between geometric inverse problems and physical meaning.

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- Linearized travel time data leads to X-ray tomography.

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- In some measurement set $U \subset M$ one can see spheres with any center.

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- With fiberwise analyticity this information can be globalized to give the universal cover of (M, F).
- The "directionality" of Finsler geometry is a major complication in comparison to the Riemannian version (de Hoop–Holman–Iversen–Lassas–Ursin, 2015).

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- One can reconstruct M and F on the good set $G \subset TM$, but not outside it. There is no such complication in Riemannian geometry (Kurylev, 1997; Katchalov–Kurylev–Lassas 2001).
- If F is fiberwise real analytic (elasticity or Riemann!), then F is determined uniquely.

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- This broken scattering relation can see much more of TM, but the trapped set is still invisible.
- Global uniqueness is can be done with added assumptions: reversibility (point symmetry) and foliation.
- Almost no assumptions are needed in the Riemannian case (Kurylev–Lassas–Uhlmann, 2010).

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- Variations in position (Q) and momentum (P) satisfy an equation

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- Written in terms of a Jacobi field J and its covariant derivative, we have instead

$$D_t \begin{pmatrix} J(t) \\ D_t J(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ -R(t) & 0 \end{pmatrix} \begin{pmatrix} J(t) \\ D_t J(t) \end{pmatrix}.$$

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- Just as differential geometry gives tools to understand how slowness surfaces vary from point to point, algebraic geometry geometry helps understand each slowness surface.
- Anisotropy helps by making the slowness surface irreducible.

Thank you!

Key ideas:

- Geometrization of geomathematics.
- Anisotropic elasticity leads to Finsler geometry.

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