The light ray transform

NCSU geometry and topology seminar

Joonas Ilmavirta

Tampere University

In collaboration with:
A. Feizmohammadi & Y. Kian & L. Oksanen

25 March 2021

Finnish Centre of Excellence in Inverse Modelling and Imaging

2018-2025

CENTRES OF EXCELLENCE IN RESEARCH

UNIVERSITY OF OULU

UNIVERSITY OF JYVÄSKYLÄ

Tampere University

FINNISH METEOROLOGICAL INSTITUTE

UNIVERSITY OF EASTERN FINLAND

LUT
Lappeenranta
University of Technology

UNIVERSITY OF HELSINKI

Aalto University

Outline

1. Light rays
   - Flat spacetimes
   - Lorentz manifolds
   - Light cones and rays
   - The light ray transform
   - The X-ray transform
2. Relation to other problems
3. Light ray tomography of scalar fields
4. Light ray tomography of tensor fields
5. Proofs

Please interrupt
at any time
in any way!

Flat spacetimes

- Flat spacetime $=$ Minkowski space $= \mathbb{R}^n$ with a certain “metric” structure.

Flat spacetimes

- Flat spacetime $=$ Minkowski space $= \mathbb{R}^n$ with a certain “metric” structure.
- In the normal Euclidean space all geometry comes from the quadratic form

$$\mathbb{R}^n \ni x \mapsto x_1^2 + x_2^2 + \cdots + x_n^2. \quad = |x|^2$$

Flat spacetimes

- Flat spacetime $=$ Minkowski space $= \mathbb{R}^n$ with a certain “metric” structure.
- In the normal Euclidean space all geometry comes from the quadratic form

$$\mathbb{R}^n \ni x \mapsto x_1^2 + x_2^2 + \cdots + x_n^2.$$

- In Minkowski spaces all geometry comes from the quadratic form

$$\mathbb{R}^n \ni x \mapsto x_1^2 - x_2^2 - \cdots - x_n^2 =: |x|^2.$$

Not really a square..

Flat spacetimes

- Flat spacetime $=$ Minkowski space $= \mathbb{R}^n$ with a certain "metric" structure.
- In the normal Euclidean space all geometry comes from the quadratic form

$$\mathbb{R}^n \ni x \mapsto x_1^2 + x_2^2 + \cdots + x_n^2.$$

- In Minkowski spaces all geometry comes from the quadratic form

$$\mathbb{R}^n \ni x \mapsto x_1^2 - x_2^2 - \cdots - x_n^2 =: |x|^2 .$$

- The coordinates $(x_2, \ldots, x_n)$ are for space, x_1 for time.

Lorentz manifolds

- Riemannian manifold: Smooth manifold M where every tangent space T_xM has a Euclidean structure (positive definite quadratic form).

Lorentz manifolds

- Riemannian manifold: Smooth manifold M where every tangent space T_xM has a Euclidean structure (positive definite quadratic form).
- Lorentzian manifold: Smooth manifold M where every tangent space T_xM has a quadratic form with eigenvalues $(+,-,\dots,-)$. or $(-,+,\dots,+)$

inner product

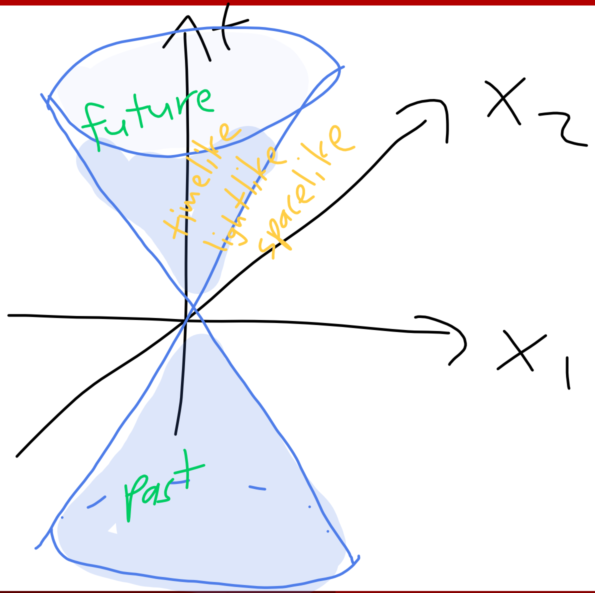
Lorentz manifolds

- Riemannian manifold: Smooth manifold M where every tangent space T_xM has a Euclidean structure (positive definite quadratic form).
- Lorentzian manifold: Smooth manifold M where every tangent space T_xM has a quadratic form with eigenvalues $(+,-,\dots,-)$.
- In both cases there is an invertible metric tensor which gives rise to connections, geodesics, and many others. But not length!

Lorentz manifolds

- Riemannian manifold: Smooth manifold M where every tangent space T_xM has a Euclidean structure (positive definite quadratic form).
- Lorentzian manifold: Smooth manifold M where every tangent space T_xM has a quadratic form with eigenvalues $(+, -, \dots, -)$.
- In both cases there is an invertible metric tensor which gives rise to connections, geodesics, and many others.
- Geodesics are straight curves, not (locally) shortest.

Light cones and rays



light cone

$$= \{v \in \mathbb{R}^{1+2}, |v|^2 = 0\}$$

No unit vectors!

Light cones and rays

A curve $\gamma : \mathbb{R} \to M$ is lightlike if $\dot\gamma(t)$ is lightlike ($|\dot\gamma(t)|^2 = 0$) for all $t \in \mathbb{R}$.

A light ray is a lightlike geodesic.

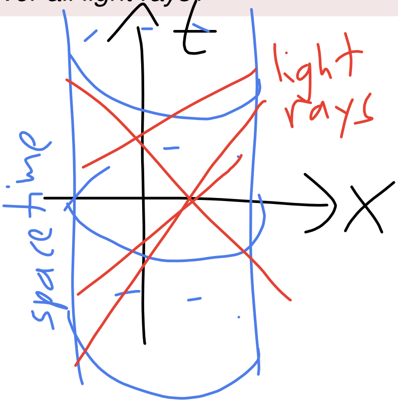
There are "constant speed" parametrizations but not unit speed.

The light ray transform

Question

Is a function $f : M \to \mathbb{R}$ uniquely determined by its integrals over all light rays?

If I know Lf, do I get f?



The light ray transform

Question

Is a function $f: M \to \mathbb{R}$ uniquely determined by its integrals over all light rays?

Definition

The light ray transform is the operator

$$L\colon \{\textit{functions on } M\} \to \{\textit{functions on the set of light rays}\}$$

$$\begin{array}{cc} \Psi & \Psi \\ f & Lf \end{array}$$

given by

$$Lf(\gamma) = \int_\gamma f.$$

Question

Is the light ray transform an injective linear operator?

The light ray transform

Question

Is the light ray transform an injective linear operator?

Question

What if f is a one-form?

The light ray transform

Question

Is the light ray transform an injective linear operator?

Question

What if f is a one-form? Or another tensor field?

The X-ray transform

On a Riemannian manifold N:

Not Lorentzian

The X-ray transform

On a Riemannian manifold N: e.g. $\mathbb{R}^n$

Definition

The X-ray transform is the operator

$$I : \{\textit{functions on } N\} \to \{\textit{functions on the set of geodesics}\}$$

not light rays

given by

$$If(\gamma) = \int_\gamma f.$$

e.g. lines

The X-ray transform

On a Riemannian manifold N:

Definition

The X-ray transform is the operator

$$I\colon \{\textit{functions on } N\} \to \{\textit{functions on the set of geodesics}\}$$

given by

$$If(\gamma) = \int_\gamma f.$$

Question

Is the X-ray transform an injective linear operator?

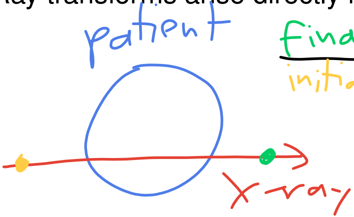
Outline

1. Light rays
2. Relation to other problems
   - Ray transforms in general
   - Wave equations
3. Light ray tomography of scalar fields
4. Light ray tomography of tensor fields
5. Proofs

Ray transforms in general

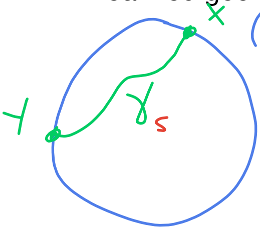
$\mu(x)$ = attenuation coefficient

- Ray transforms arise directly in imaging applications, e.g. CT.



$$\frac{\text{final intensity}}{\text{initial intensity}} = \exp\Big(\underbrace{\int_\gamma \mu}_{\text{X-ray transform}}\Big)$$

- Ray transforms arise directly in imaging applications, e.g. CT.
- Linearized geometric problems often lead to ray transforms.



(M, g_s)

$$\frac{d}{ds} d_s(x,y)\Big|_{s=0} \approx \int_{\gamma_0} \overbrace{\left(\frac{d}{ds} g_s\Big|_{s=0}\right)}^{=:f}$$

Ray transforms in general

- Ray transforms arise directly in imaging applications, e.g. CT.
- Linearized geometric problems often lead to ray transforms.
- Inverse problems for PDEs can be turned to ray transform problems if one can focus solutions on a ray.

Wave equations

- The usual wave equation in $\mathbb{R}^{1+n}$ is

$$\underbrace{[\partial_0^2 - \partial_1^2 - \cdots - \partial_n^2]}_{=:\Box \text{ usually}} u(t,x) = 0.$$

Wave equations

- The usual wave equation in $\mathbb{R}^{1+n}$ is
$$[\partial_0^2 - \partial_1^2 - \cdots - \partial_n^2]u(t,x) = 0.$$

- The wave operator on $\mathbb{R}^{1+n}$ is the Laplace–Beltrami operator of the Minkowski space.

Δ_g

Wave equations

- The usual wave equation in $\mathbb{R}^{1+n}$ is

$$[\partial_0^2 - \partial_1^2 - \cdots - \partial_n^2]u(t,x) = 0.$$

- The wave operator on $\mathbb{R}^{1+n}$ is the Laplace–Beltrami operator of the Minkowski space.
- Solutions to the wave equation can be non-smooth.

$1+1D$: $u(t,x) = \delta(t-x)$

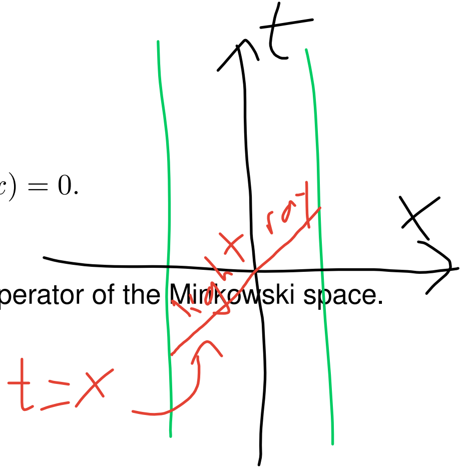
Wave equations

- The usual wave equation in $\mathbb{R}^{1+n}$ is

$$[\partial_0^2 - \partial_1^2 - \cdots - \partial_n^2]u(t,x) = 0.$$

- The wave operator on $\mathbb{R}^{1+n}$ is the Laplace–Beltrami operator of the Minkowski space.
- Solutions to the wave equation can be non-smooth.
- Singularities of solutions follow light rays.

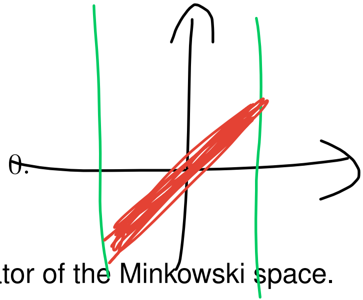
$u(t,x) = \delta(t-x)$



Wave equations

- The usual wave equation in $\mathbb{R}^{1+n}$ is

$$[\partial_0^2 - \partial_1^2 - \cdots - \partial_n^2]u(t,x) = 0.$$



- The wave operator on $\mathbb{R}^{1+n}$ is the Laplace–Beltrami operator of the Minkowski space.
- Solutions to the wave equation can be non-smooth.
- Singularities of solutions follow light rays.
- Wave packets or other asymptotic solutions can be built around light rays.

Wave equations

- On a Lorentzian manifold (M, g) the wave equation is

$$\Delta_g u(x) = 0.$$

Now x is an event (in spacetime), not a point (in space).

Wave equations

- On a Lorentzian manifold (M, g) the wave equation is

$$\Delta_g u(x) = 0.$$

  Now x is an event (in spacetime), not a point (in space).
- To this equation you can add a scalar potential (a function $q\colon M \to \mathbb{R}$ is a potential):

$$[\Delta_g + q(x)]u(x) = 0.$$

Wave equations

- On a Lorentzian manifold (M, g) the wave equation is

2nd order operator

$$\Delta_g u(x) = 0.$$

  Now x is an event (in spacetime), not a point (in space).
- To this equation you can add a scalar potential (a function $q: M \to \mathbb{R}$ is a potential):

0th order term

$$[\Delta_g + q(x)]u(x) = 0.$$

- Or a vector potential (a vector field A on M):

1st order term

$$[-(-i\nabla_g + A)^2 + q]u = 0.$$

Wave equations

Consider the Lorentzian manifold $M = \mathbb{R} \times \Omega$, where Ω is compact (e.g. $\subset \mathbb{R}^n$).

Wave equations

Consider the Lorentzian manifold $M = \mathbb{R} \times \Omega$, where Ω is compact (e.g. $\subset \mathbb{R}^n$).

Definition

The Cauchy data of q and A is

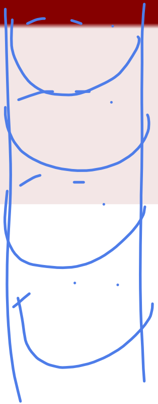
all solutions

$$C(q, A) = \{(u|_{\partial M}, \partial_\nu u|_{\partial M}); [(-i\nabla_g + A)^2 + q]u = 0\}.$$

(This is the graph of the Dirichlet-to-Neumann operator.)

boundary measurements

$$\Lambda_{q,A} : u|_{\partial M} \mapsto \partial_\nu u|_{\partial M}$$



Wave equations

Consider the Lorentzian manifold $M = \mathbb{R} \times \Omega$, where Ω is compact (e.g. $\subset \mathbb{R}^n$).

Definition

The Cauchy data of q and A is

$$C(q, A) = \{(u|_{\partial M}, \partial_\nu u|_{\partial M}); [(-i\nabla_g + A)^2 + q]u = 0\}.$$

(This is the graph of the Dirichlet-to-Neumann operator.)

Question

Suppose q_i and A_i are compactly supported in M. If $C(q_1, A_1) = C(q_2, A_2)$, is $q_1 = q_2$ and $A_1 = A_2$?

Wave equations

Consider the Lorentzian manifold $M = \mathbb{R} \times \Omega$, where Ω is compact (e.g. $\subset \mathbb{R}^n$).

Definition

The Cauchy data of q and A is

$$C(q, A) = \{(u|_{\partial M}, \partial_\nu u|_{\partial M}); [(-i\nabla_g + A)^2 + q]u = 0\}.$$

(This is the graph of the Dirichlet-to-Neumann operator.)

Question

Suppose q_i and A_i are compactly supported in M. If $C(q_1, A_1) = C(q_2, A_2)$, is $q_1 = q_2$ and $A_1 = A_2$?

No, because $C(q, A) = C(q, A + \nabla\phi)$ for any scalar function ϕ with zero boundary values.

But is this all?

Wave equations

Lemma

Suppose $C(q_1, A_1) = C(q_2, A_2)$. *Then for any light ray* γ *through* M

$$\int_\gamma (q_1 - q_2)\mathrm{d}t = L\overbrace{(q_1 - q_2)}^{\text{scalar field}}(\gamma) = 0$$

and

$$\int_\gamma (A_1 - A_2) = L\overbrace{(A_1 - A_2)}^{\text{vector field}}(\gamma) = 0.$$

Wave equations

Lemma

Suppose $C(q_1, A_1) = C(q_2, A_2)$*. Then for any light ray* γ *through* M

$$\int_\gamma (q_1 - q_2)\mathrm{d}t = L(q_1 - q_2)(\gamma) = 0$$

and

$$\int_\gamma (A_1 - A_2) = L(A_1 - A_2)(\gamma) = 0.$$

If L is injective on scalar and vector fields, then the Cauchy data determines the two potentials!

... modulo some obstructions.

Outline

Flat spacetimes

Take any $n \geq 2$ and let $\mathbb{R}^{1+n}$ be the Minkowski space of dimension $1+n$.

Flat spacetimes

Take any $n \geq 2$ and let $\mathbb{R}^{1+n}$ be the Minkowski space of dimension $1+n$.

Theorem

Let $f: \mathbb{R}^{1+n} \to \mathbb{C}$ be a compactly supported smooth function. If $Lf(\gamma) = 0$ for all γ, then $f = 0$.

Flat spacetimes

Take any $n \geq 2$ and let $\mathbb{R}^{1+n}$ be the Minkowski space of dimension $1+n$.

Theorem

Let $f : \mathbb{R}^{1+n} \to \mathbb{C}$ be a compactly supported smooth function. If $Lf(\gamma) = 0$ for all γ, then $f = 0$.

That is, the light ray transform is injective on $C_c^\infty(\mathbb{R}^{1+n})$.

Flat spacetimes

Take any $n \geq 2$ and let $\mathbb{R}^{1+n}$ be the Minkowski space of dimension $1+n$.

Theorem

Let $f : \mathbb{R}^{1+n} \to \mathbb{C}$ be a compactly supported smooth function. If $Lf(\gamma) = 0$ for all γ, then $f = 0$.

That is, the light ray transform is injective on $C_c^\infty(\mathbb{R}^{1+n})$.

But there is a Schwartz function $f \neq 0$ for which $Lf = 0$!

Many, and we know which ones.

Static spacetimes

Static spacetimes

A stationary spacetime admits a product structure $M = \mathbb{R} \times N$ and the metric tensor is conformal to

$$g = \mathrm{d}t^2 + \mathrm{d}t \otimes \eta(x) + \eta(x) \otimes \mathrm{d}t - h(x),$$

where h is a Riemannian metric on N and η is a one-form on N.

Static spacetimes

A stationary spacetime admits a product structure $M = \mathbb{R} \times N$ and the metric tensor is conformal to

$$g = \mathrm{d}t^2 \cancel{+ \mathrm{d}t \otimes \eta(x) + \eta(x) \otimes \mathrm{d}t} - h(x),$$

where h is a Riemannian metric on N and η is a one-form on N.

The spacetime is called static if $\eta = 0$. This is full product geometry:

$$\text{spacetime} = \text{time} \times \text{space}.$$

Riemannian product: $g = g_1 + g_2$

Lorentzian product. $g = g_1 - g_2$

Static spacetimes

A stationary spacetime admits a product structure $M = \mathbb{R} \times N$ and the metric tensor is conformal to

$$g = \mathrm{d}t^2 + \mathrm{d}t \otimes \eta(x) + \eta(x) \otimes \mathrm{d}t - h(x),$$

where h is a Riemannian metric on N and η is a one-form on N.

The spacetime is called static if $\eta = 0$. This is full product geometry:

$$\text{spacetime} = \text{time} \times \text{space}.$$

The Minkowski space is static (and thus stationary).

Feel free to keep thinking Minkowski!

Theorem (Feizmohammadi–J.I.–Kian–Oksanen & Feizmohammadi–J.I.–Oksanen)

Static spacetimes

Theorem (Feizmohammadi–J.I.–Kian–Oksanen & Feizmohammadi–J.I.–Oksanen)

Consider a static spacetime $M = \mathbb{R} \times N$, where N is a compact Riemannian manifold with boundary.

Static spacetimes

Theorem (Feizmohammadi–J.I.–Kian–Oksanen & Feizmohammadi–J.I.–Oksanen)

Consider a static spacetime $M = \mathbb{R} \times N$, where N is a compact Riemannian manifold with boundary.

If the Riemannian X-ray transform is injective on N, then the light ray transform is injective on (compactly supported functions on) M.

Injectivity is inherited

Riemann $\rightsquigarrow$ Lorentz

Static spacetimes

Theorem (Feizmohammadi–J.I.–Kian–Oksanen & Feizmohammadi–J.I.–Oksanen)

Consider a static spacetime $M = \mathbb{R} \times N$, where N is a compact Riemannian manifold with boundary.

If the Riemannian X-ray transform is injective on N, then the light ray transform is injective on (compactly supported functions on) M.

Works also in stationary geometry!

whatever that might mean...

Outline

How to integrate a tensor field

A (covariant) tensor field f of rank m gives rise to a multilinear map

$$f_x \colon T_xM \times \cdots \times T_xM \to \mathbb{R}$$

at each $x \in M$.

How to integrate a tensor field

A (covariant) tensor field f of rank m gives rise to a multilinear map

$$f_x\colon T_xM \times \cdots \times T_xM \to \mathbb{R}$$

at each $x \in M$.

The metric tensor is a rank 2 covariant tensor field.

How to integrate a tensor field

A (covariant) tensor field f of rank m gives rise to a multilinear map

$$f_x\colon T_xM \times \cdots \times T_xM \to \mathbb{R}$$

at each $x \in M$.

The metric tensor is a rank 2 covariant tensor field.

The integral of a tensor field f along a curve $\gamma\colon [a,b] \to M$ is

$$\int_\gamma f = \int_a^b f_{\gamma(t)}(\dot\gamma(t),\dots,\dot\gamma(t))\mathrm{d}t.$$

How to integrate a tensor field

A (covariant) tensor field f of rank m gives rise to a multilinear map

$$f_x \colon T_xM \times \cdots \times T_xM \to \mathbb{R}$$

at each $x \in M$.

The metric tensor is a rank 2 covariant tensor field.

The integral of a tensor field f along a curve $\gamma \colon [a,b] \to M$ is

$$\int_\gamma f = \int_a^b f_{\gamma(t)}(\dot\gamma(t), \ldots, \dot\gamma(t))\mathrm{d}t.$$

If $m=0$, $f_{\gamma(t)} =: f(\gamma(t))$

This gives the familiar formulas when $m = 0, 1$.

Potential kernel

Question

If a tensor field f integrates to zero over all light rays, is $f = 0$?

Potential kernel

Question

If a tensor field f integrates to zero over all light rays, is $f = 0$?

No! Unless $m = 0$

Potential kernel

Question

If a tensor field f integrates to zero over all light rays, is $f = 0$?

No!

Case $m = 1$: If $f = \mathrm{d}h$ where h is a scalar function vanishing on the boundary, then

$$\int_\gamma f = h(\gamma(t_{\mathrm{end}})) - h(\gamma(t_{\mathrm{start}})) = 0$$

for any light ray γ.

Potential kernel

Question

If a tensor field f integrates to zero over all light rays, is $f = 0$?

No!

Case $m = 1$: If $f = \mathrm{d}h$ where h is a scalar function vanishing on the boundary, then

$$\int_\gamma f = h(\gamma(t_{\text{end}})) - h(\gamma(t_{\text{start}})) = 0$$

covariant derivative

for any light ray γ.

symmetrization

Case $m \geq 1$: If h is a tensor field of rank $m-1$ vanishing on the boundary and $f = \sigma \nabla h$, then $Lf = 0$.

Vector field tomography

Theorem (Riemannian geometry)

Vector field tomography

Theorem (Riemannian geometry)

Let N be a simple Riemannian manifold.

- convex boundary
- unique geodesics

e.g. $\overline{B}(0,1) \subset \mathbb{R}^n$

Vector field tomography

Theorem (Riemannian geometry)

Let N be a simple Riemannian manifold. The following are equivalent for a covector field (= one-form = covariant tensor field of rank 1) f on N:

1. *f integrates to zero over all geodesics.*
2. *$f = \mathrm{d}h$ for some function $h\colon N \to \mathbb{R}$ with $h|_{\partial N} = 0$.*

Vector field tomography

Theorem (Riemannian geometry)

Let N be a simple Riemannian manifold. The following are equivalent for a covector field (= one-form = covariant tensor field of rank 1) f on N:

1. f *integrates to zero over all geodesics.*
2. $f = \mathrm{d}h$ *for some function* $h \colon N \to \mathbb{R}$ *with* $h|_{\partial N} = 0$.

Theorem (Feizmohammadi–J.I.–Kian–Oksanen & Feizmohammadi–J.I.–Oksanen)

Vector field tomography

Theorem (Riemannian geometry)

Let N be a simple Riemannian manifold. The following are equivalent for a covector field (= one-form = covariant tensor field of rank 1) f on N:

1. *f integrates to zero over all geodesics.*
2. *$f = \mathrm{d}h$ for some function $h\colon N \to \mathbb{R}$ with $h|_{\partial N} = 0$.*

Theorem (Feizmohammadi–J.I.–Kian–Oksanen & Feizmohammadi–J.I.–Oksanen)

Consider a static spacetime $M = \mathbb{R} \times N$, where N is a simple Riemannian manifold.

Vector field tomography

Theorem (Riemannian geometry)

Let N be a simple Riemannian manifold. The following are equivalent for a covector field (= one-form = covariant tensor field of rank 1) f on N:

1. *f integrates to zero over all geodesics.*
2. *$f = \mathrm{d}h$ for some function $h\colon N \to \mathbb{R}$ with $h|_{\partial N} = 0$.*

Theorem (Feizmohammadi–J.I.–Kian–Oksanen & Feizmohammadi–J.I.–Oksanen)

Consider a static spacetime $M = \mathbb{R} \times N$, where N is a simple Riemannian manifold.

The following are equivalent for a compactly supported covector field f on M:

1. *f integrates to zero over all light rays.*
2. *$f = \mathrm{d}h$ for some function $h\colon M \to \mathbb{R}$ with $h|_{\partial M} = 0$.*

Conformal and antisymmetric kernel

Conformal and antisymmetric kernel

The integral of a tensor field f along a curve $\gamma\colon [a,b] \to M$ is

$$\int_\gamma f = \int_a^b f_{\gamma(t)}(\dot\gamma(t),\dots,\dot\gamma(t))\mathrm{d}t.$$

Conformal and antisymmetric kernel

The integral of a tensor field f along a curve $\gamma\colon [a,b] \to M$ is

$$\int_\gamma f = \int_a^b f_{\gamma(t)}(\dot\gamma(t), \dots, \dot\gamma(t)) \mathrm{d}t.$$

These integrals only see the symmetric part of f!

Conformal and antisymmetric kernel

The integral of a tensor field f along a curve $\gamma\colon [a,b] \to M$ is

$$\int_\gamma f = \int_a^b f_{\gamma(t)}(\dot\gamma(t),\dots,\dot\gamma(t))\mathrm{d}t.$$

These integrals only see the symmetric part of f!

For $m = 2$ we can write $f = f_{\text{symmetric}} + f_{\text{antisymmetric}}$ and $Lf_{\text{antisymmetric}} = 0$.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

Conformal and antisymmetric kernel

The integral of a tensor field f along a curve $\gamma\colon [a,b] \to M$ is

$$\int_\gamma f = \int_a^b f_{\gamma(t)}(\dot\gamma(t),\dots,\dot\gamma(t))\mathrm{d}t.$$

These integrals only see the symmetric part of f!

For $m = 2$ we can write $f = f_{\text{symmetric}} + f_{\text{antisymmetric}}$ and $Lf_{\text{antisymmetric}} = 0$.

Ray transforms are often only defined for symmetric tensor fields.

Conformal and antisymmetric kernel

The integral of a tensor field f along a curve $\gamma\colon [a,b] \to M$ is

$$\int_\gamma f = \int_a^b f_{\gamma(t)}(\dot\gamma(t),\ldots,\dot\gamma(t))\mathrm{d}t.$$

These integrals only see the symmetric part of f!

For $m = 2$ we can write $f = f_{\text{symmetric}} + f_{\text{antisymmetric}}$ and $Lf_{\text{antisymmetric}} = 0$.

Ray transforms are often only defined for symmetric tensor fields.

The potential kernel and this antisymmetric kernel exist for any kinds of rays.

geodesics, light rays, ...

Conformal and antisymmetric kernel

Light rays are geodesics for which $|\dot\gamma(t)|^2 = 0$ (or $g(\dot\gamma(t), \dot\gamma(t)) = 0$) for all t.

Conformal and antisymmetric kernel

Light rays are geodesics for which $|\dot{\gamma}(t)|^2 = 0$ (or $g(\dot{\gamma}(t), \dot{\gamma}(t)) = 0$) for all t.

The light ray transform of the metric tensor is zero:

$$Lg(\gamma) = \int_a^b \underbrace{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}_{\equiv 0!} \mathrm{d}t = 0.$$

Conformal and antisymmetric kernel

Light rays are geodesics for which $|\dot\gamma(t)|^2 = 0$ (or $g(\dot\gamma(t), \dot\gamma(t)) = 0$) for all t.

The light ray transform of the metric tensor is zero:

$$Lg(\gamma) = \int_a^b g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))\mathrm{d}t = 0.$$

If $f = cg$ for any scalar function c, then $Lf = 0$.

Conformal and antisymmetric kernel

Light rays are geodesics for which $|\dot\gamma(t)|^2 = 0$ (or $g(\dot\gamma(t), \dot\gamma(t)) = 0$) for all t.

The light ray transform of the metric tensor is zero:

$$Lg(\gamma) = \int_a^b g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))\mathrm{d}t = 0.$$

If $f = cg$ for any scalar function c, then $Lf = 0$.

tensor product

Case $m \geq 2$: If c is a tensor field of rank $m-2$, then $L(c \otimes g) = 0$.

Conformal and antisymmetric kernel

A tensor field f of rank $m \geq 2$ is in the kernel of the light ray transform if:

- $f = \nabla h$ for h of rank $m-1$, potential fields
- $f = c \otimes g$ for c of rank $m-2$, or conformal fields
- f is "antisymmetric".

OG to fully symmetric

$\neq$ fully antisymmetric if $m \geq 3$

Conformal and antisymmetric kernel

A tensor field f of rank $m \geq 2$ is in the kernel of the light ray transform if:

- $f = \nabla h$ for h of rank $m - 1$,
- $f = c \otimes g$ for c of rank $m - 2$, or
- f is "antisymmetric".

Conjecture

$Lf = 0$ if and only if the symmetric part of f is of the form

$$\sigma(\nabla h + c \otimes g).$$

force symmetry

Light ray tensor tomography

A Riemannian manifold N is nice if a symmetric tensor field f integrates to zero (if and) only if $f = \sigma\nabla h$ and $h|_{\partial N} = 0$.

Light ray tensor tomography

A Riemannian manifold N is nice if a symmetric tensor field f integrates to zero (if and) only if $f = \sigma\nabla h$ and $h|_{\partial N} = 0$. There are nice manifolds, e.g. Euclidean domains.

Light ray tensor tomography

A Riemannian manifold N is nice if a symmetric tensor field f integrates to zero (if and) only if $f = \sigma\nabla h$ and $h|_{\partial N} = 0$. There are nice manifolds, e.g. Euclidean domains.

Theorem (Feizmohammadi–J.I.–Oksanen)

Suppose N is nice and let $M = \mathbb{R} \times N$.

Light ray tensor tomography

A Riemannian manifold N is nice if a symmetric tensor field f integrates to zero (if and) only if $f = \sigma\nabla h$ and $h|_{\partial N} = 0$. There are nice manifolds, e.g. Euclidean domains.

Theorem (Feizmohammadi–J.I.–Oksanen)

Suppose N is nice and let $M = \mathbb{R} \times N$. The following are equivalent for a tensor field f of rank m on M:

1. $Lf = 0$, *meaning that f integrates to zero over all light rays.*
2. $f_{\mathrm{sym}} = \sigma(\nabla h + c \otimes g)$ *for some tensor fields h of rank $m-1$ and c of rank $m-2$ with $h|_{\partial M} = 0$.*

Conformal symmetry

Conformal symmetry

A conformal transformation is $g \mapsto cg$, where c is a scalar function.

$c(x) > 0 \;\; \forall x \in M$

Conformal symmetry

A conformal transformation is $g \mapsto cg$, where c is a scalar function.

Theorem

Light rays as sets are invariant under conformal transformations.

But parametrizations change!

Conformal symmetry

A conformal transformation is $g \mapsto cg$, where c is a scalar function.

Theorem

Light rays as sets are invariant under conformal transformations.

Theorem (Feizmohammadi–J.I.–Oksanen)

Injectivity of the light ray transform up to natural obstructions is invariant under conformal transformations.

- potential fields
- conformal fields
- symmetry

Conformal symmetry

A conformal transformation is $g \mapsto cg$, where c is a scalar function.

Theorem

Light rays as sets are invariant under conformal transformations.

Theorem (Feizmohammadi–J.I.–Oksanen)

Injectivity of the light ray transform up to natural obstructions is invariant under conformal transformations.

Both the question and the answer are conformally invariant in ~~some~~ sense.

different!

Outline

Minkowski geometry

Minkowski geometry

Minkowski sense

- Take any $v \in \mathbb{R}^{1+n}$ with $|v|^2 = 0$.

Minkowski geometry

- Take any $v \in \mathbb{R}^{1+n}$ with $|v|^2 = 0$.
- Take any $\xi \in \mathbb{R}^{1+n}$ with $\xi \perp v$.

M.S.

Minkowski geometry

- Take any $v \in \mathbb{R}^{1+n}$ with $|v|^2 = 0$.
- Take any $\xi \in \mathbb{R}^{1+n}$ with $\xi \perp v$.
- The function $x \cdot \xi$ is invariant when x is translated in the direction of v, so

Fubini

M. S.

$$\int_{\mathbb{R}^{1+n}} e^{-ix\cdot\xi} f(x)$$

can be written in terms of of the light ray transform.

different exp.
coefficient on
each ray

light rays

Minkowski geometry

- Take any $v \in \mathbb{R}^{1+n}$ with $|v|^2 = 0$.
- Take any $\xi \in \mathbb{R}^{1+n}$ with $\xi \perp v$.
- The function $x \cdot \xi$ is invariant when x is translated in the direction of v, so

  $$\int_{\mathbb{R}^{1+n}} e^{-ix\cdot\xi} f(x)$$

  can be written in terms of of the light ray transform.
- If $Lf = 0$, then $\hat{f}(\xi) = 0$, provided that ξ has an orthogonal lightlike vector v.

Minkowski geometry

- Take any $v \in \mathbb{R}^{1+n}$ with $|v|^2 = 0$.
- Take any $\xi \in \mathbb{R}^{1+n}$ with $\xi \perp v$.
- The function $x \cdot \xi$ is invariant when x is translated in the direction of v, so

$$\int_{\mathbb{R}^{1+n}} e^{-ix\cdot\xi} f(x)$$

  can be written in terms of of the light ray transform.
- If $Lf = 0$, then $\hat{f}(\xi) = 0$, provided that ξ has an orthogonal lightlike vector v. Thus $\hat{f}(\xi) = 0$ whenever $|\cancel{\xi^2}| \leq 0$.

$|\xi|^2$ Spacelike

Minkowski geometry

- Take any $v \in \mathbb{R}^{1+n}$ with $|v|^2 = 0$.
- Take any $\xi \in \mathbb{R}^{1+n}$ with $\xi \perp v$.
- The function $x \cdot \xi$ is invariant when x is translated in the direction of v, so

$$\int_{\mathbb{R}^{1+n}} e^{-ix\cdot\xi} f(x)$$

  can be written in terms of of the light ray transform.
- If $Lf = 0$, then $\hat{f}(\xi) = 0$, provided that ξ has an orthogonal lightlike vector v. Thus $\hat{f}(\xi) = 0$ whenever $\left|\xi^2\right| \leq 0$.
- If f is compactly supported, $\hat{f}$ is real analytic.

Paley-Wiener

Minkowski geometry

- Take any $v \in \mathbb{R}^{1+n}$ with $|v|^2 = 0$.
- Take any $\xi \in \mathbb{R}^{1+n}$ with $\xi \perp v$.
- The function $x \cdot \xi$ is invariant when x is translated in the direction of v, so

$$\int_{\mathbb{R}^{1+n}} e^{-ix\cdot\xi} f(x)$$

  can be written in terms of of the light ray transform.
- If $Lf = 0$, then $\hat{f}(\xi) = 0$, provided that ξ has an orthogonal lightlike vector v. Thus $\hat{f}(\xi) = 0$ whenever $\left|\xi^2\right| \leq 0$.
- If f is compactly supported, $\hat{f}$ is real analytic. Now $\hat{f} = 0$ in an open set, so $\hat{f} = 0$ everywhere.

Minkowski geometry

- Take any $v \in \mathbb{R}^{1+n}$ with $|v|^2 = 0$.
- Take any $\xi \in \mathbb{R}^{1+n}$ with $\xi \perp v$.
- The function $x \cdot \xi$ is invariant when x is translated in the direction of v, so

$$\int_{\mathbb{R}^{1+n}} e^{-ix\cdot\xi} f(x)$$

  can be written in terms of of the light ray transform.
- If $Lf = 0$, then $\hat{f}(\xi) = 0$, provided that ξ has an orthogonal lightlike vector v. Thus $\hat{f}(\xi) = 0$ whenever $\left|\xi^2\right| \leq 0$.
- If f is compactly supported, $\hat{f}$ is real analytic. Now $\hat{f} = 0$ in an open set, so $\hat{f} = 0$ everywhere. Thus $f = 0$.

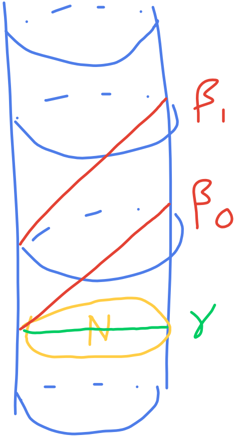
Product geometry

Product geometry

- Light rays look like $t \mapsto (s+t, \gamma(t)) =: \beta_s(t)$, where γ is a Riemannian geodesic.

Product geometry

- Light rays look like $t \mapsto (s+t, \gamma(t)) =: \beta_s(t)$, where γ is a Riemannian geodesic. Fix any γ.



Product geometry

- Light rays look like $t \mapsto (s+t, \gamma(t)) =: \beta_s(t)$, where γ is a Riemannian geodesic. Fix any γ.
- We start from $Lf(\beta_s) = 0$.

Product geometry

- Light rays look like $t \mapsto (s+t, \gamma(t)) =: \beta_s(t)$, where γ is a Riemannian geodesic. Fix any γ.
- We start from $Lf(\beta_s) = 0$.
- Fourier transform in s:

$$0 = \int_{\mathbb{R}} e^{-i\tau s} Lf(\beta_s)\mathrm{d}s = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\tau s} f(s+t,\gamma(t))\mathrm{d}t\mathrm{d}s = \int_{\mathbb{R}} e^{i\tau t}\hat{f}(\tau,\gamma(t))\mathrm{d}t.$$

$\forall \tau \in \mathbb{R}$

Product geometry

- Light rays look like $t \mapsto (s+t, \gamma(t)) =: \beta_s(t)$, where γ is a Riemannian geodesic. Fix any γ.
- We start from $Lf(\beta_s) = 0$.
- Fourier transform in s:

FT in time only!

$$0 = \int_{\mathbb{R}} e^{-i\tau s} Lf(\beta_s)\mathrm{d}s = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\tau s} f(s+t, \gamma(t))\mathrm{d}t\mathrm{d}s = \int_{\mathbb{R}} e^{i\tau t} \hat{f}(\tau, \gamma(t))\mathrm{d}t.$$

- At $\tau = 0$: The Riemannian X-ray transform of $\hat{f}(0, \cdot\,)$ vanishes, so $\hat{f}(0, \cdot\,) = 0$.

Product geometry

- Light rays look like $t \mapsto (s+t, \gamma(t)) =: \beta_s(t)$, where γ is a Riemannian geodesic. Fix any γ.

$\cancel{it\hat{f}(0,\gamma(t))} + \partial_\tau \hat{f}(0,\gamma(t))$

- We start from $Lf(\beta_s) = 0$.
- Fourier transform in s:

$$0 = \int_{\mathbb{R}} e^{-i\tau s} Lf(\beta_s) \mathrm{d}s = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\tau s} f(s+t, \gamma(t)) \mathrm{d}t \mathrm{d}s = \int_{\mathbb{R}} e^{i\tau t} \hat{f}(\tau, \gamma(t)) \mathrm{d}t.$$

- At $\tau = 0$: The Riemannian X-ray transform of $\hat{f}(0, \cdot\,)$ vanishes, so $\hat{f}(0, \cdot\,) = 0$.
- Derivative at $\tau = 0$: The Riemannian X-ray transform of $\partial_\tau \hat{f}(\tau, \cdot\,)|_{\tau=0}$ vanishes, so it is zero.

Product geometry

- Light rays look like $t \mapsto (s+t, \gamma(t)) =: \beta_s(t)$, where γ is a Riemannian geodesic. Fix any γ.
- We start from $Lf(\beta_s) = 0$.
- Fourier transform in s:

$\partial_\tau^k \hat{f} + 0 + 0 + \cdots$

∂_τ^k at $\tau = 0$

$$0 = \int_{\mathbb{R}} e^{-i\tau s} Lf(\beta_s) \mathrm{d}s = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\tau s} f(s+t, \gamma(t)) \mathrm{d}t \mathrm{d}s = \int_{\mathbb{R}} e^{i\tau t} \hat{f}(\tau, \gamma(t)) \mathrm{d}t.$$

- At $\tau = 0$: The Riemannian X-ray transform of $\hat{f}(0, \cdot\,)$ vanishes, so $\hat{f}(0, \cdot\,) = 0$.
- Derivative at $\tau = 0$: The Riemannian X-ray transform of $\partial_\tau \hat{f}(\tau, \cdot\,)|_{\tau=0}$ vanishes, so it is zero.
- Iterate for all orders and the Taylor series in τ is zero.

Product geometry

- Light rays look like $t \mapsto (s+t, \gamma(t)) =: \beta_s(t)$, where γ is a Riemannian geodesic. Fix any γ.
- We start from $Lf(\beta_s) = 0$.
- Fourier transform in s:

$$0 = \int_{\mathbb{R}} e^{-i\tau s} Lf(\beta_s)\mathrm{d}s = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\tau s} f(s+t, \gamma(t))\mathrm{d}t\mathrm{d}s = \int_{\mathbb{R}} e^{i\tau t} \hat{f}(\tau, \gamma(t))\mathrm{d}t.$$

- At $\tau = 0$: The Riemannian X-ray transform of $\hat{f}(0, \cdot\,)$ vanishes, so $\hat{f}(0, \cdot\,) = 0$.
- Derivative at $\tau = 0$: The Riemannian X-ray transform of $\partial_\tau \hat{f}(\tau, \cdot\,)|_{\tau=0}$ vanishes, so it is zero.
- Iterate for all orders and the Taylor series in τ is zero. By analyticity $\hat{f} = 0$ and so $f = 0$.

Paley-Wiener

Thank you!

Key ideas:

- Light rays.
- Inverse problems for wave-like equations.
- Injectivity of the light ray transform.
- Kernel characterization for tensor fields.

`http://users.jyu.fi/~jojapeil`
`joonas.ilmavirta@tuni.fi`