

# The light ray transform

NCSU geometry and topology seminar

Joonas Ilmavirta

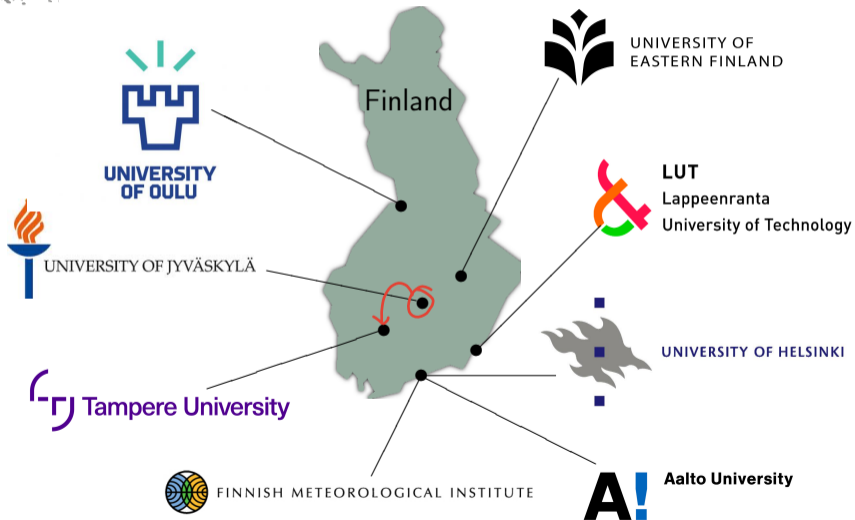
Tampere University

In collaboration with:

A. Feizmohammadi & Y. Kian & L. Oksanen

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# Outline

- 1 Light rays
  - Flat spacetimes
  - Lorentz manifolds
  - Light cones and rays
  - The light ray transform
  - The X-ray transform
- 2 Relation to other problems
- 3 Light ray tomography of scalar fields
- 4 Light ray tomography of tensor fields
- 5 Proofs

Please interrupt  
at any time  
in any way!

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*Not really a square ..*

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- The coordinates  $(x_2, \dots, x_n)$  are for space,  $x_1$  for time.

- Riemannian manifold: Smooth manifold  $M$  where every tangent space  $T_x M$  has a Euclidean structure (positive definite quadratic form).



# Lorentz manifolds

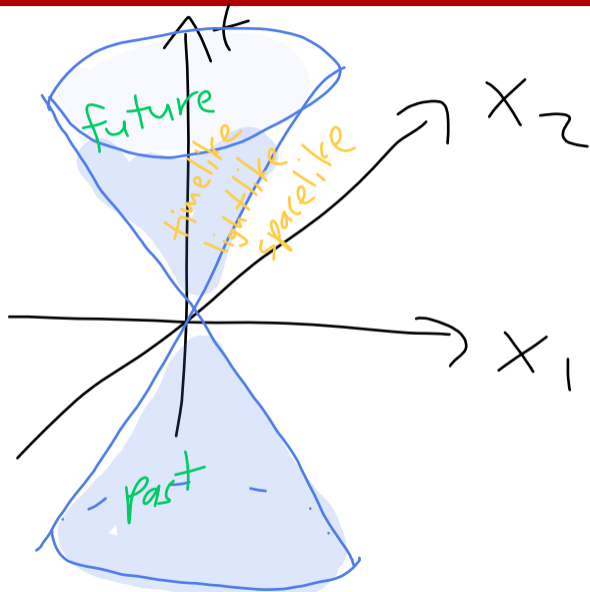
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inner product

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- In both cases there is an invertible metric tensor which gives rise to connections, geodesics, and many others.
- Geodesics are straight curves, not (locally) shortest.

# Light cones and rays



light cone  
 $\{v \in \mathbb{R}^{1+2}, |v|^2 = 0\}$   
No unit vectors!

## Light cones and rays

A curve  $\gamma: \mathbb{R} \rightarrow M$  is lightlike if  $\dot{\gamma}(t)$  is lightlike ( $|\dot{\gamma}(t)|^2 = 0$ ) for all  $t \in \mathbb{R}$ .

A light ray is a lightlike geodesic.

# The light ray transform

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*Is a function  $f: M \rightarrow \mathbb{R}$  uniquely determined by its integrals over all light rays?*

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## Definition

*The light ray transform is the operator*

$$L: \{\text{functions on } M\} \rightarrow \{\text{functions on the set of light rays}\}$$

*given by*

$$Lf(\gamma) = \int_{\gamma} f.$$

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- 1 Light rays
- 2 Relation to other problems
  - Ray transforms in general
  - Wave equations
- 3 Light ray tomography of scalar fields
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- Linearized geometric problems often lead to ray transforms.
- Inverse problems for PDEs can be turned to ray transform problems if one can focus solutions on a ray.

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- Solutions to the wave equation can be non-smooth.
- Singularities of solutions follow light rays.
- Wave packets or other asymptotic solutions can be built around light rays.

# Wave equations

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$$[\Delta_g + q(x)]u(x) = 0.$$

- Or a vector potential (a vector field  $A$  on  $M$ ):

$$[-(-i\nabla_g + A)^2 + q]u = 0.$$

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$$C(q, A) = \{(u|_{\partial M}, \partial_\nu u|_{\partial M}); [(-i\nabla_g + A)^2 + q]u = 0\}.$$

*(This is the graph of the Dirichlet-to-Neumann operator.)*

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Suppose  $q_i$  and  $A_i$  are compactly supported in  $M$ . If  $C(q_1, A_1) = C(q_2, A_2)$ , is  $q_1 = q_2$  and  $A_1 = A_2$ ?

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No, because  $C(q, A) = C(q, A + \nabla\phi)$  for any scalar function  $\phi$  with zero boundary values.

## Lemma

Suppose  $C(q_1, A_1) = C(q_2, A_2)$ . Then for any light ray  $\gamma$  through  $M$

$$\int_{\gamma} (q_1 - q_2) dt = L(q_1 - q_2)(\gamma) = 0$$

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If  $L$  is injective on scalar and vector fields, then the Cauchy data determines the two potentials!



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## Theorem

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But there is a Schwartz function  $f \neq 0$  for which  $Lf = 0$ !

# Static spacetimes

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A stationary spacetime admits a product structure  $M = \mathbb{R} \times N$  and the metric tensor is conformal to

$$g = dt^2 + dt \otimes \eta(x) + \eta(x) \otimes dt - h(x),$$

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The Minkowski space is static (and thus stationary).

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*Consider a static spacetime  $M = \mathbb{R} \times N$ , where  $N$  is a compact Riemannian manifold with boundary.*

*If the Riemannian X-ray transform is injective on  $N$ , then the light ray transform is injective on (compactly supported functions on)  $M$ .*

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Works also in stationary geometry!

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- 4 Light ray tomography of tensor fields
  - How to integrate a tensor field
  - Potential kernel
  - Vector field tomography
  - Conformal and antisymmetric kernel
  - Light ray tensor tomography
  - Conformal symmetry
- 5 Proofs

# How to integrate a tensor field

A (covariant) tensor field  $f$  of rank  $m$  gives rise to a multilinear map

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This gives the familiar formulas when  $m = 0, 1$ .

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Case  $m = 1$ : If  $f = dh$  where  $h$  is a scalar function vanishing on the boundary, then

$$\int_{\gamma} f = h(\gamma(t_{\text{end}})) - h(\gamma(t_{\text{start}})) = 0$$

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Case  $m \geq 1$ : If  $h$  is a tensor field of rank  $m - 1$  vanishing on the boundary and  $f = \sigma \nabla h$ , then  $Lf = 0$ .

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The potential kernel and this antisymmetric kernel exist for any kinds of rays.

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Case  $m \geq 2$ : If  $c$  is a tensor field of rank  $m - 2$ , then  $L(c \otimes g) = 0$ .

# Conformal and antisymmetric kernel

A tensor field  $f$  of rank  $m \geq 2$  is in the kernel of the light ray transform if:

- $f = \nabla h$  for  $h$  of rank  $m - 1$ ,
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## Conjecture

$Lf = 0$  if and only if the symmetric part of  $f$  is of the form

$$\sigma(\nabla h + c \otimes g).$$



A Riemannian manifold  $N$  is nice if a symmetric tensor field  $f$  integrates to zero (if and) only if  $f = \sigma \nabla h$  and  $h|_{\partial N} = 0$ .

# Light ray tensor tomography

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## Theorem (Feizmohammadi–J.I.–Oksanen)

Suppose  $N$  is nice and let  $M = \mathbb{R} \times N$ . The following are equivalent for a tensor field  $f$  of rank  $m$  on  $M$ :

- 1  $Lf = 0$ , meaning that  $f$  integrates to zero over all light rays.
- 2  $f_{\text{sym}} = \sigma(\nabla h + c \otimes g)$  for some tensor fields  $h$  of rank  $m - 1$  and  $c$  of rank  $m - 2$  with  $h|_{\partial M} = 0$ .

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Both the question and the answer are conformally invariant in some sense.

- 1 Light rays
- 2 Relation to other problems
- 3 Light ray tomography of scalar fields
- 4 Light ray tomography of tensor fields
- 5 Proofs
  - Minkowski geometry
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# Thank you!

Key ideas:

- Light rays.
- Inverse problems for wave-like equations.
- Injectivity of the light ray transform.
- Kernel characterization for tensor fields.

`http://users.jyu.fi/~jojapeil`  
`joonas.ilmavirta@tuni.fi`