Stable reconstruction of simple Riemannian manifolds from unknown interior sources

Inverse problems and nonlinearity

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Outline

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Can we tell what the manifold (the planet) has inside?
At least approximately?

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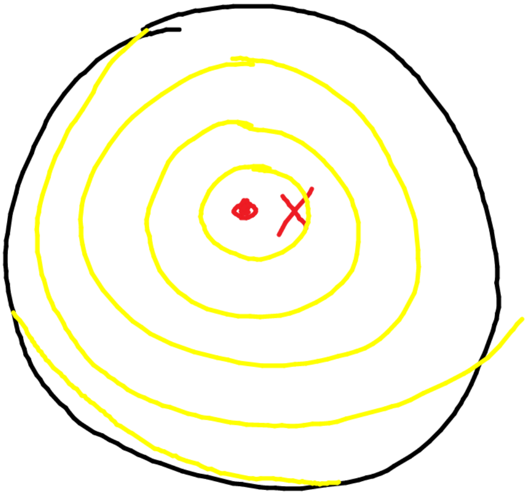

A point $x \in M$ sends waves.

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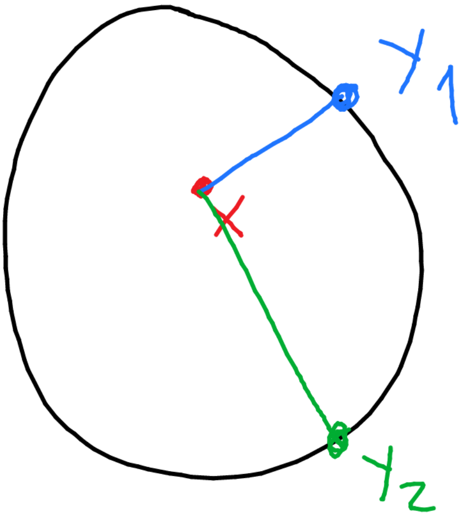

Arrival times from x are measured at y_1 and y_2.

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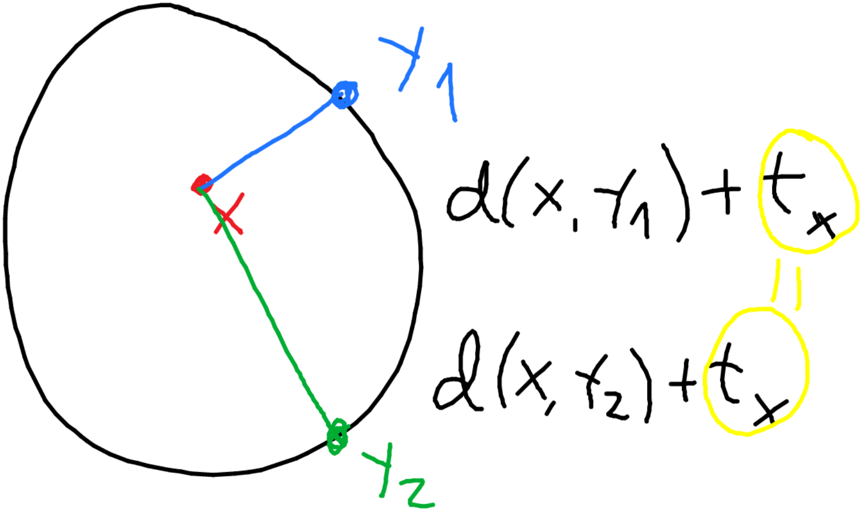

The data contains the distances offset by the time the source at x went off.

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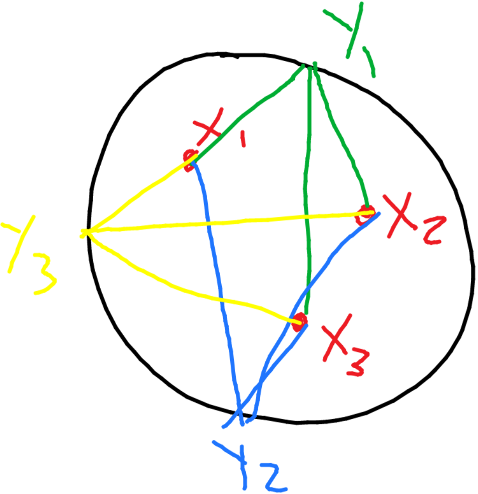

x_i goes off at time t_i

Multiple sources measured from multiple stations. Offsets in time depend on x_i but not y_j.

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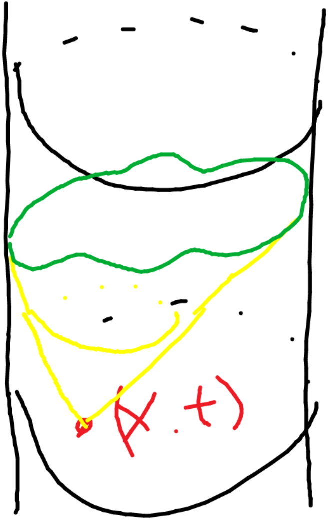

observation for (x,t) is light cone intersected with boundary

$$Q(S) \subset \partial M \times \mathbb{R}$$

Sources and measurements in the spacetime.

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The data is the set of all arrival times without any labels:

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Question

To what extend does $Q(S)$ determine M?

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- Approximate in what sense?
- If we approximate M as a metric space (Gromov–Hausdorff), then how is this reconstruction attached to the known boundary?

Outline

1. The problem
2. Related problems
   - Sources on the boundary
   - Interior sources with additional data
3. Labeled Gromov–Hausdorff distance
4. Theorems

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- Sources everywhere on the boundary: *boundary rigidity*. (Michel '81, Gromov '83, Croke '91, Pestov–Uhlmann '05, Stefanov–Uhlmann '05, Burago–Ivanov '10)

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- Known origin times, neatly separated sources everywhere: *boundary distance data*. (Kurylev '97, de Hoop-I–Lassas–Saksala '19)
- Unknown origin times, neatly separated sources everywhere: *boundary distance difference data*. (Lassas–Saksala '19, de Hoop–Saksala '19, Ivanov '20)

Outline

Hausdorff distance

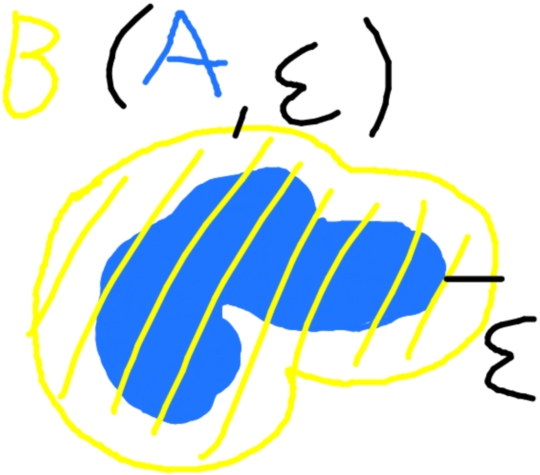
The Hausdorff distance is a distance between two compact subsets of a metric space.

Hausdorff distance

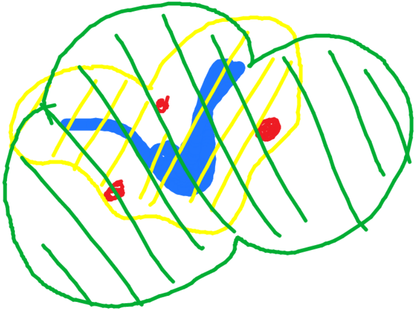
$A \subset \mathbb{R}^2$

A compact set in the plane.

Hausdorff distance



A ball of radius ε around the set.



$$\left.\begin{aligned} A &\subset B(A', \varepsilon) \\ A' &\subset B(A, \varepsilon) \end{aligned}\right\} \Rightarrow d_H(A, A') \le \varepsilon$$

Two sets within each other's balls.

Hausdorff distance

The Hausdorff distance between $A, A' \subset \mathbb{R}^2$ is the infimum of those εs so that they are contained in each other's ε-balls.

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The Gromov–Hausdorff distance is a distance between two compact metric spaces.
(No fixed ambient geometry.)

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- Consider the Hausdorff distance $d_{H,Z}(f(X), g(Y))$ within the space Z.
- The infimum (over all choices of Z) of these Hausdorff distances is the Gromov–Hausdorff distance $d_{GH}(X, Y)$.

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- The labeled Gromov–Hausdorff distance between (X, α) and (Y, β) is

$$d^L_{GH}(X, \alpha; Y, \beta) = \inf\{d_{H,Z}(f(X), g(Y)) + \sup_{\ell \in L} d_Z(f(\alpha(\ell)), g(\beta(\ell)));$$

$$Z \text{ is a compact metric space},$$

$$f\colon X \to Z \text{ and } g\colon Y \to Z \text{ are isometric embeddings}\}.$$

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is small.
- Then the approximate manifold is constructed in some relation to the known boundary, not floating unattached.

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- The labeled Gromov–Hausdorff distance is a metric:
  1. Symmetry.
  2. Triangle inequality.
  3. $d_{GH}^{L}(X, \alpha; Y, \beta) = 0$ if and only if there is an isometry $h \colon X \to Y$ so that $h \circ \alpha = \beta$.

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A Riemannian manifold satisfies our estimates if and only if it is simple.

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This holds on many but not all simple manifolds.

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We can also get some information on the distance to boundary points, but nothing perfect.

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where the error ε is explicit.

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From the data and a priori geometric bounds we get an approximate finite model and an estimate on the reconstruction error!

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If we measure for an increasing amount of time, we get an increasingly good approximate reconstruction.

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Myers–Steenrod: Metric isometry implies smooth isometry.

Thank you!

Summary:

- Unknown source points, unknown origin times, unlabeled arrival time data.
- Approximate reconstruction in labeled Gromov–Hausdorff distance.
- Perfect reconstruction in infinite or increasing time.
- arXiv:2102.11799

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