Eigenfrequencies: Spectral rigidity with discontinuities

Geo-Mathematical Imaging Group
Project Review

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Eigenfrequencies: Spectral rigidity with discontinuities

spectrum of frequencies

$\downarrow$

trace

$\downarrow$

singularities

$\downarrow$

length spectrum

$\downarrow$

radial velocity profile

Outline

The spectrum of eigenfrequencies

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- The amplitudes of the various modes depend on the event, the frequencies do not.
- The set of frequencies is the spectrum of free oscillations.
- These frequencies are (square roots of) eigenvalues of an elliptic partial differential operator, “the elastic Laplacian”.

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- The function $t \mapsto \mathrm{tr}(\partial_t G)$ is mostly smooth. It has a singularity at $t = T$ if and pretty much only if T is the length of a periodic ray path (periodic broken geodesic).

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- Using the length spectral data is like travel time tomography but without endpoints.

Linearization

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- Linearizing normal travel time tomography between surface points leads to geodesic X-ray tomography.
  - This problem is well understood on some manifolds.
- Linearizing periodic travel time tomography leads to periodic broken ray tomography.
  - This problem is beginning to be understood on very few manifolds.

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$\approx$ Earth-like planets are locally uniquely determined by the spectrum of free oscillations.

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- These are equivalent — up to changing coordinates.

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- The geodesic flow is integrable with conserved quantities.
- Geometry behaves well under the Herglotz condition: $\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r}{c(r)}\right) > 0$ for all $r > 0$.

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- The results hold for the isotropic elastic system (2 radial Lamé parameters) but are trickier to state.
- Waves split 4 ways at interfaces.
- In the scalar case the spectrum of free oscillations is modeled geometrically “simply” by the Neumann spectrum of a Laplace–Beltrami operator.

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- it stays in a single layer,
- it stays in the same polarization, and
- it is of one of two types:
  - It only reflects from the top interface.
  - It is radial.

The lengths of these constitute the basic length spectrum.

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- Conjugate points cause trouble in geometric proofs!
- We make two assumptions:
  - Countable conjugacy condition:
    There are only countably many conjugate points (up to rotations and shifts).
  - Periodic conjugacy condition:
    No point is self-conjugate along a basic ray.

The theorems

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A parameter $\tau \in (-\varepsilon, \varepsilon)$ describing a family of models.

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The spectral or length spectral data determines the interfaces and the radial velocities in a linearized sense.

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- Is Herglotz a consequence of physics?

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- In the high frequency limit the amplitudes and the relative surface amplitudes go down.
- Noise is a problem.

Outline

1. The idea
2. Mathematics
3. Physics
4. The Herglotz condition
   - The classical Herglotz condition
   - Reduced phase space
   - Herglotz as convexity

The classical Herglotz condition

$$\rho(r) := \frac{r}{c(r)}$$

$$\rho'(r) > 0$$

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- Properties:
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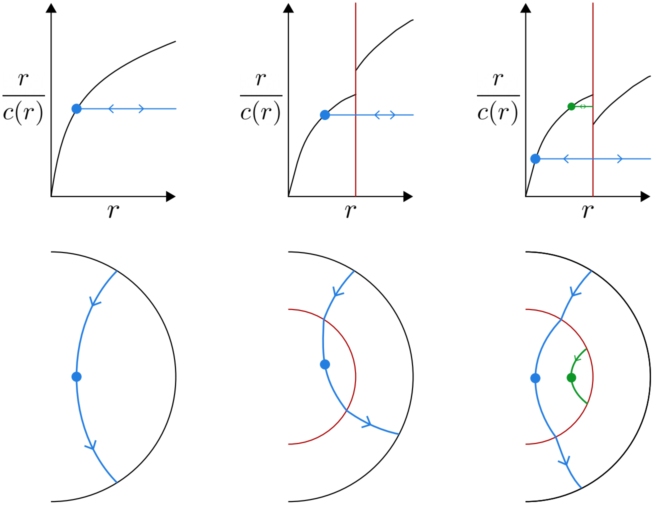
  - $L \leq \rho(r)$ everywhere.
  - $L = \rho(r)$ if and only if the radial component of the velocity is zero ($\dot{r} = 0$).

Reduced phase space



The reduced phase space is under the graph of ρ.

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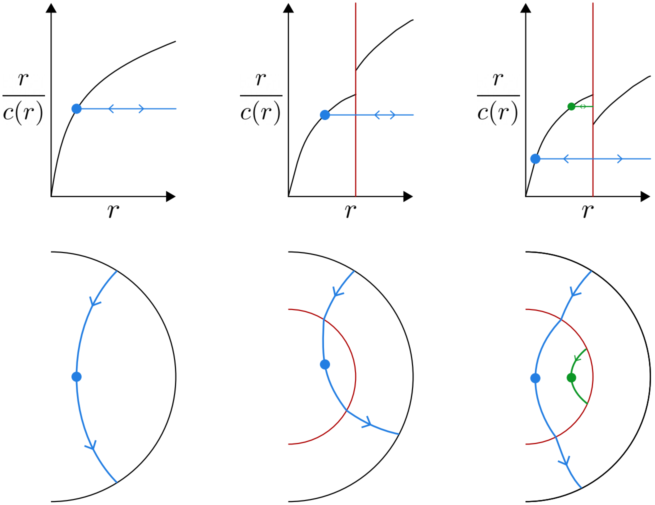
- In the presence of discontinuities: Herglotz in the sense of distributions means convexity.
- For our theorems we only need Herglotz in every layer. Jumps can be in any direction.

Herglotz as convexity



All of these are fine for us.

Thank you!

Key ideas:

- Spectrum determines length spectrum through a trace formula.
- Jump discontinuities are not a problem.
- *Should* be implementable.

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