Geometric inverse problems arising from geophysics

UCL inverse problems seminar

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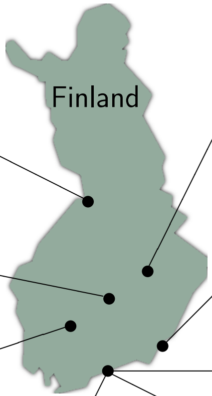
CENTRES OF EXCELLENCE IN RESEARCH



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Geometric inverse problems arising from geophysics

physics

↓

PDE

↓

geometry

↓

inverse problems

↓

physics

Outline

Newton's theory

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- The gravitational force exerted by the Sun causes the Earth's trajectory to curve.
- The force is described by a simple formula and the equation of motion is an ODE in $\mathbb{R}^n$.
- The Newtonian approach is straightforward to use and often a good model.

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- There is a complicated equation of motion for the geometry itself: Einstein's field equation is a non-linear system of coupled PDEs.
- This model is harder to use but can reach phenomena inaccessible to Newtonian gravity and provides a more geometric way to see the essential structures.

The goal

A geometric theory of elasticity?

Untoy the model: Bring mathematics closer to the application.

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- Density normalized: $a_{ijkl}(x) = c_{ijkl}(x)/\rho(x)$.

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- If the material is anisotropic (c is no more symmetric than necessary and wave speed depends on direction), then the vector nature of the equation cannot be ignored.
- Elastic waves arising from earthquakes (or marsquakes!) satisfy this equation away from the focus of the event to great accuracy. (Weak field limit.)

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- The principal symbol of the EWO is $\Gamma(x,\xi) - \omega^2 I$, where $\xi = \omega p$.

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- Decomposition to polarizations only works on the level of singularities. The individual polarizations do not satisfy PDEs.

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- The admissible slowness vectors p are on the slowness surface given by the equation

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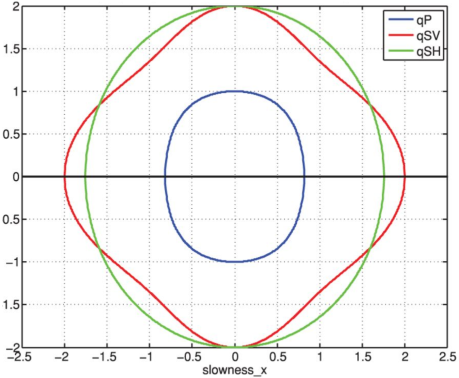

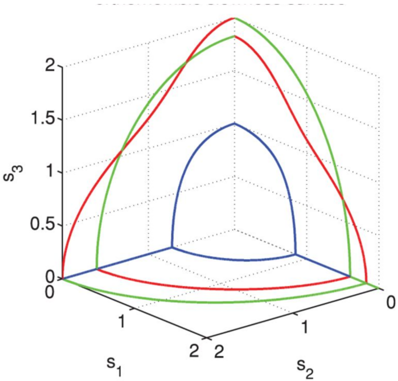

The slowness surface. Smaller slowness $\iff$ faster wave.

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- There are two geometries: “spatial” and “temporal”.

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- Compare to gravitation!

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- In elastic geometry we measure distance in travel time, and the waves go straight in this geometry.
- Fermat’s principle: Phonons — the particles corresponding to elastic waves — go straight in the geometry given by travel time.
- Fermat’s principle is about going straight in the relevant geometry, not about taking the shortest path. These are not the same thing over long distances or for shear waves.

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- The norms on the dual spaces T_x^*M satisfy the same conditions.

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Other polarizations are problematic.

Outline

1. Geometrization of gravitation
2. Elastic waves
3. Elastic geometry
4. Inverse problems
   - Geometrization of the question
   - Herglotz (Mönkkönen)
   - Dix (de Hoop, Lassas)
   - Distance function (de Hoop, Lassas, Saksala)
   - Scattering data (de Hoop, Lassas, Saksala)
   - Ray tracing (Iversen, Ursin, Saksala, de Hoop)
   - And more. . .

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- Practical goal for a geometer: Given some boundary data, find the Finsler manifold — or its cosphere bundle.

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- Linearized travel time data leads to X-ray tomography.

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- The "directionality" of Finsler geometry is a major complication in comparison to the Riemannian version (de Hoop–Holman–Iversen–Lassas–Ursin, 2015).

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- If F is fiberwise real analytic (elasticity or Riemann!), then F is determined uniquely.

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- This broken scattering relation can see much more of TM, but the trapped set is still invisible.
- Global uniqueness is can be done with added assumptions: reversibility (point symmetry) and foliation.
- Almost no assumptions are needed in the Riemannian case (Kurylev–Lassas–Uhlmann, 2010).

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- Written in terms of a Jacobi field J and its covariant derivative, we have instead

$$D_t \begin{pmatrix} J(t) \\ D_t J(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ -R(t) & 0 \end{pmatrix} \begin{pmatrix} J(t) \\ D_t J(t) \end{pmatrix}.$$

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Modelling goal

Building a complete theory of elastic geometry.

Thank you!

Key ideas:

- Pure mathematics for the sake of physics.
- Geometrization.
- Geomathematics.

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