Geometric inverse problems arising from geophysics

Jyväskylä analysis seminar

Joonas Ilmavirta

Tampere University

9 Sep 2020

Positions

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- –July 2020: Senior researcher at Jyväskylä

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Tampere

Tampere University of Technology

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University of Tampere

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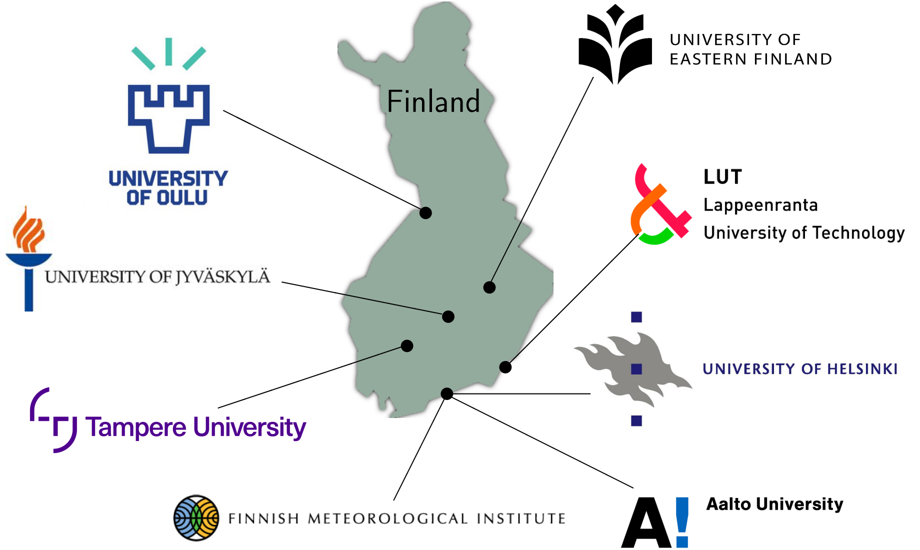
Tampere University

Finnish Centre of Excellence in Inverse Modelling and Imaging

2018-2025

CENTRES OF EXCELLENCE IN RESEARCH



Geometric inverse problems arising from geophysics

physics

$\downarrow$

PDE

$\downarrow$

geometry

$\downarrow$

inverse problems

$\downarrow$

physics

Outline

Inverse modelling and imaging

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The goal

I want to make sure that the mathematical results I prove about imaging and inverse problems correspond well to physical models.

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I will give two examples.

Elastic geometry

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Enough generality to allow for known physical phenomena and maybe a little more.

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I am studying geometric neutrino data with Gunther Uhlmann.

Outline

1. Mathematical modelling
2. Geometrization of gravitation
   - Newton's theory
   - Einstein's theory
   - The goal
3. Elastic waves
4. Elastic geometry
5. Inverse problems

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- The gravitational force exerted by the Sun causes the Earth's trajectory to curve.
- The force is described by a simple formula and the equation of motion is an ODE in $\mathbb{R}^n$.
- The Newtonian approach is straightforward to use and often a good model.

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- There is a complicated equation of motion for the geometry itself: Einstein's field equation is a non-linear system of coupled PDEs.
- This model is harder to use but can reach phenomena inaccessible to Newtonian gravity and provides a more geometric way to see the essential structures.

The goal

A geometric theory of elasticity?

Untoy the model: Bring mathematics closer to the application.

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- Density normalized: $a_{ijkl}(x) = c_{ijkl}(x)/\rho(x)$.

The elastic wave equation

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- Using Newton's second law with a restoring force given by Hooke's law leads to the elastic wave equation (EWE)

$$\partial_j[c_{ijkl}(x)\partial_k u_l(x,t)] - \rho(x)\partial_t^2 u_i(x,t) = 0,$$

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  where $u(x,t)$ is a small displacement field.
- If the material is anisotropic (c is no more symmetric than necessary and wave speed depends on direction), then the vector nature of the equation cannot be ignored.
- Elastic waves arising from earthquakes (or marsquakes!) satisfy this equation away from the focus of the event to great accuracy.

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- The principal symbol of the EWO is $\Gamma(x,\xi) - \omega^2 I$, where $\xi = \omega p$.

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- Polarization vectors are eigenvectors of the Christoffel matrix Γ, so they are orthogonal.
- Decomposition to polarizations only works on the level of singularities. The individual polarizations do not satisfy PDEs.

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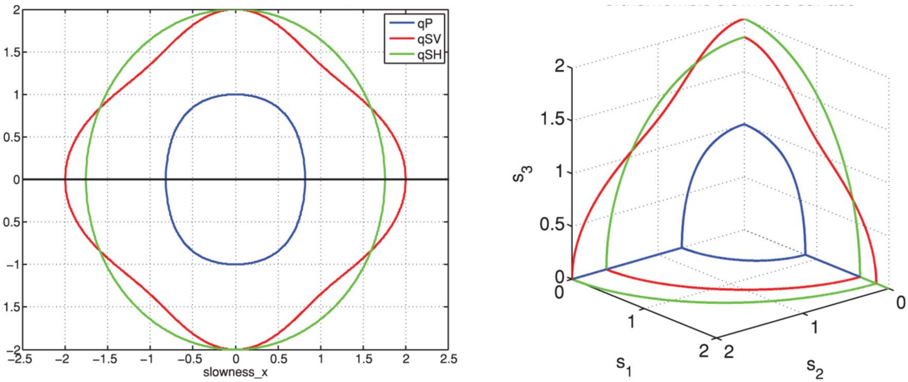
- The slowness vector p and the polarization A of a singularity at x must satisfy
$$[\Gamma(x,p) - I]A = 0.$$
- The admissible slowness vectors p are on the slowness surface given by the equation
$$\det(\Gamma(x,p) - I) = 0.$$

Singularities and the slowness surface



The slowness surface. Smaller slowness $\iff$ faster wave.

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1. Mathematical modelling
2. Geometrization of gravitation
3. Elastic waves
4. Elastic geometry
   - Distance
   - Ray tracing
   - Finsler manifolds
5. Inverse problems

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- Distance is measured in units of time.

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  - Traditional view: The trajectory of the phonon is curved because wave speed varies.
  - Newer view: The phonon goes straight in a curved geometry (geodesic), and the geometry is curved by variations in wave speed.

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- In elastic geometry we measure distance in travel time, and the waves go straight in this geometry.
- Fermat’s principle: Phonons — the particles corresponding to elastic waves — go straight in the geometry given by travel time.
- Fermat’s principle is about going straight in the relevant geometry, not about taking the shortest path. These are not the same thing over long distances or for shear waves.

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- The norms on the dual spaces $T_x^* M$ satisfy the same conditions.

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Other polarizations are problematic.

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5. Inverse problems
   - Geometrization of the question
   - Herglotz (Mönkkönen)
   - Dix (de Hoop, Lassas)
   - Distance function (de Hoop, Lassas, Saksala)
   - Scattering data (de Hoop, Lassas, Saksala)
   - Ray tracing (Iversen, Ursin, Saksala, de Hoop)
   - And more…

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- Practical goal for a geometer: Given some boundary data, find the Finsler manifold — or its cosphere bundle.

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- There is still a Herglotz condition but it looks different.
- Linearized travel time data leads to X-ray tomography.

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- With fiberwise analyticity this information can be globalized to give the universal cover of (M, F).

Dix (de Hoop, Lassas)

- Imagine that we have (virtual) point sources everywhere in the planet and we can measure the arriving wave fronts. Does this determine the reduced stiffness tensor a everywhere without any kind of isotropy assumption?
- To geometrize the problem, consider a Finsler manifold (M, F).
- In some measurement set $U \subset M$ one can see spheres with any center. The data consists of oriented surfaces with radii.
- One can follow the geodesics backwards and find the metric on a neighborhood of the lift.
- With fiberwise analyticity this information can be globalized to give the universal cover of (M, F).
- The "directionality" of Finsler geometry is a major complication in comparison to the Riemannian version (de Hoop–Holman–Iversen–Lassas–Ursin, 2015).

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- One can reconstruct M and F on the good set $G \subset TM$, but not outside it. There is no such complication in Riemannian geometry (Kurylev, 1997; Katchalov–Kurylev–Lassas 2001).
- If F is fiberwise real analytic (elasticity or Riemann!), then F is determined uniquely.

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- This broken scattering relation can see much more of TM, but the trapped set is still invisible.
- Global uniqueness is can be done with added assumptions: reversibility (point symmetry) and foliation.
- Almost no assumptions are needed in the Riemannian case (Kurylev–Lassas–Uhlmann, 2010).

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- Written in terms of a Jacobi field J and its covariant derivative, we have instead

$$D_t \begin{pmatrix} J(t) \\ D_t J(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ -R(t) & 0 \end{pmatrix} \begin{pmatrix} J(t) \\ D_t J(t) \end{pmatrix}.$$

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Modelling goal

Building a complete theory of elastic geometry.

Thank you!

Key ideas:

- Pure mathematics for the sake of physics.
- Geometrization.
- Geomathematics.

http://users.jyu.fi/~jojapeil

joonas.ilmavirta@tuni.fi