Towards geometrization of geophysics

Inverse Days 2020

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- A geometric theory of elasticity?

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- In elastic geometry
  - the object is an elastic body like the Earth and
  - the distance is the travel time of seismic waves.

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- We can then study how these particles travel.
  - Traditional view: The trajectory of a phonon is curved because wave speed varies.
  - Newer view: A phonon goes straight in a curved geometry (along a geodesic), and the geometry is curved by variations in wave speed.

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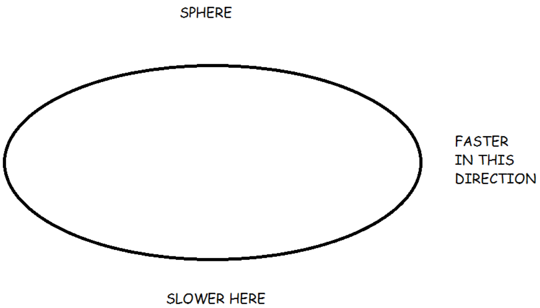

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- The cosphere (the slowness surface) describes the reciprocal of phase velocity.

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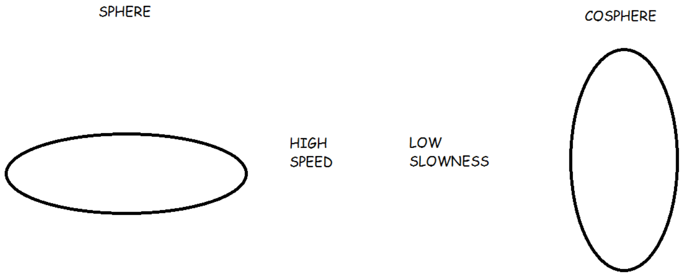

Sphere and cosphere, anisotropic.

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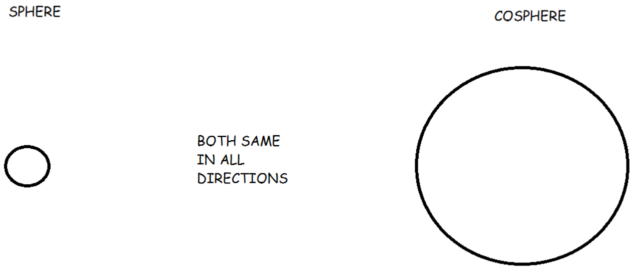

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- Elliptic anisotropy = the sphere and cosphere are ellipsoids.

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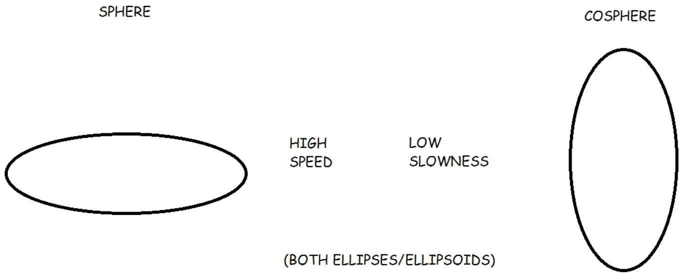

Sphere and cosphere, elliptically anisotropic.

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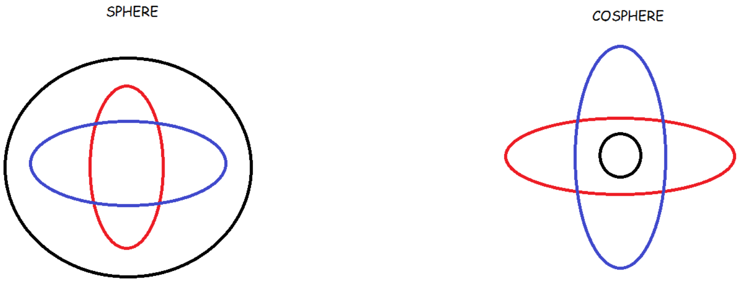

Three polarizations, all elliptically anisotropic.

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- If the cosphere is not convex, the sphere can branch.

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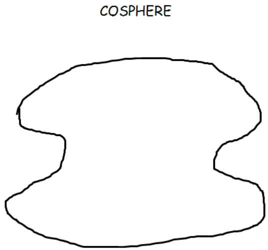
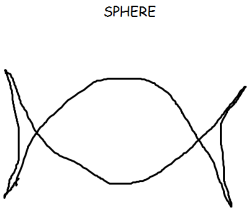
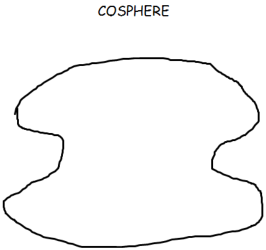


A non-convex cosphere.

Anisotropy





A branched sphere.

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$$\omega^2[\Gamma(x,p) - I],$$

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- The slowness surface consists of those slowness vectors $p \in T_x^*\mathbb{R}^3$ where the symbol is not invertible:

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- It is an algebraic variety in $T_x^*\mathbb{R}^3$, parametrized by $a_{ijkl}(x)$.

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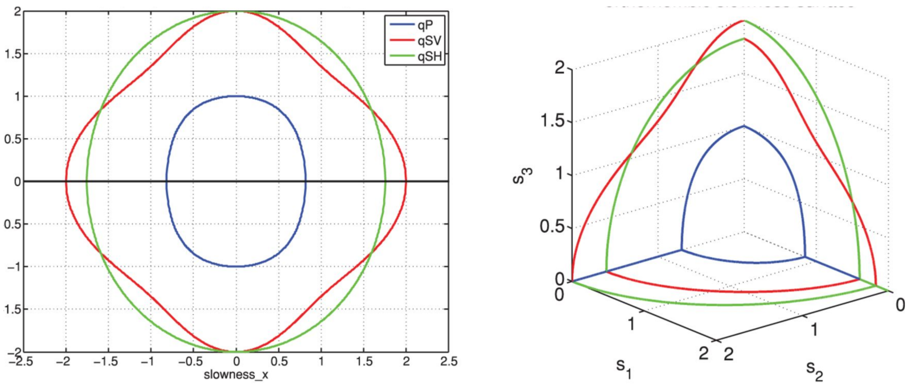

The slowness surface. Smaller slowness $\iff$ faster wave.

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- General anistropy and inhomogeneity (when convex) $\subsetneq$ Finsler manifolds.

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- A geometrized reformulation:
  Given some boundary data, find the elastic Finsler manifold
  (the cosphere or slowness surface at every point).

Conclusion

- Observation: Phonons follow Finsler geodesics.
- Inverse problems goal: Solve geometric inverse problems on Finsler manifolds.
- Modelling goal: Understand the differential and algebraic geometry of the elastic manifold.

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