Geometric inverse problems arising from geophysics

Seminar at Helsinki

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physics

$\downarrow$

PDE

$\downarrow$

geometry

$\downarrow$

inverse problems

$\downarrow$

physics

Outline

Inverse modelling and imaging

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The goal

I want to make sure that the mathematical results I prove about imaging and inverse problems correspond well to physical models.

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Enough generality to allow for known physical phenomena and maybe a little more.

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I am studying geometric neutrino data with Gunther Uhlmann.

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- Gravitation is a force and a force causes acceleration.
- The gravitational force exerted by the Sun causes the Earth's trajectory to curve.
- The force is described by a simple formula and the equation of motion is an ODE in $\mathbb{R}^n$.
- The Newtonian approach is straightforward to use and often a good model.

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- There is a complicated equation of motion for the geometry itself: Einstein’s field equation is a non-linear system of coupled PDEs.
- This model is harder to use but can reach phenomena inaccessible to Newtonian gravity and provides a more geometric way to see the essential structures.

The goal

A geometric theory of elasticity?

Untoy the model: Bring mathematics closer to the application.

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- Using Newton's second law with a restoring force given by Hooke's law leads to the elastic wave equation (EWE)

$$\partial_j [c_{ijkl}(x) \partial_k u_l(x,t)] - \rho(x) \partial_t^2 u_i(x,t) = 0,$$

where $u(x,t)$ is a small displacement field.

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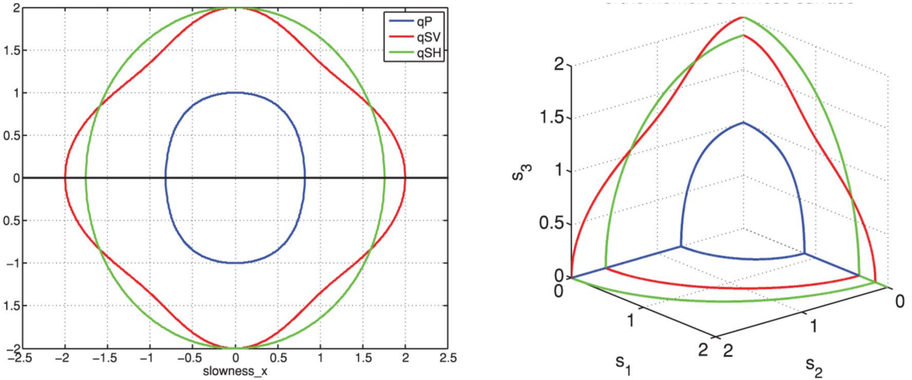
- The wave fronts (singularities) of solutions to the EWE follow the Hamiltonian flow determined by the principal symbol $\omega^2(\Gamma - I)$.
- To have $[\Gamma(x,p) - I]A = 0$ for $A \neq 0$, the slowness vector p must satisfy

$$\det(\Gamma(x,p) - I) = 0.$$

  These set of these ps is the slowness surface at x.

Microlocal analysis



The slowness surface. Smaller slowness $\iff$ faster wave.
The fastest one (quasi-pressure) is best behaved.

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- That is, we declare the distance between x and y to be the shortest amount of time it takes for a wave to travel from x to y.
- This definition only sees the fastest polarization (qP), corresponding to the innermost branch of the slowness surface. Other polarizations are tricky.

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  - Traditional view: The trajectory of the phonon is curved because wave speed varies.
  - Newer view: The phonon goes straight in a curved geometry (geodesic), and the geometry is curved by variations in wave speed.

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- The norms on the dual spaces T_x^*M satisfy the same conditions.

Finsler manifolds

Elastic finsler geometry

The cosphere (dual unit sphere) of the elastic geometry is the slowness surface. The “elastic manifold” corresponding to qP waves is a Finsler manifold with special properties. Phonons go straight in this geometry.

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Other polarizations are problematic.

Outline

1. Mathematical modelling
2. Geometrization of gravitation
3. Elastic waves and geometry
4. Inverse problems
    - Geometrization of the question
    - Distance function (de Hoop, Lassas, Saksala)
    - Scattering data (de Hoop, Lassas, Saksala)
    - And more…

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- Typical inverse problem: Given some boundary data, find the reduced stiffness tensor $a_{ijkl}(x)$ everywhere.
- A more geometric formulation: Given some boundary data, find the cosphere (slowness surface) at every point.
- Practical goal for a geometer: Given some boundary data, find the Finsler manifold — or its cosphere bundle.

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- One can reconstruct M and F on the good set $G \subset TM$, but not outside it. There is no such complication in Riemannian geometry (Kurylev, 1997; Katchalov–Kurylev–Lassas 2001).
- If F is fiberwise real analytic (elasticity or Riemann!), then F is determined uniquely.

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- This broken scattering relation can see much more of TM, but the trapped set is still invisible.
- Global uniqueness is can be done with added assumptions: reversibility (point symmetry) and foliation.
- Almost no assumptions are needed in the Riemannian case (Kurylev–Lassas–Uhlmann, 2010).

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Modelling goal

Building a complete theory of elastic geometry.

Thank you!

Key ideas:

- Pure mathematics for the sake of physics.
- Geometrization.
- Geomathematics.

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