Algebraic and differential geometry of slowness surfaces

Geo-Mathematical Imaging Group
Project Review

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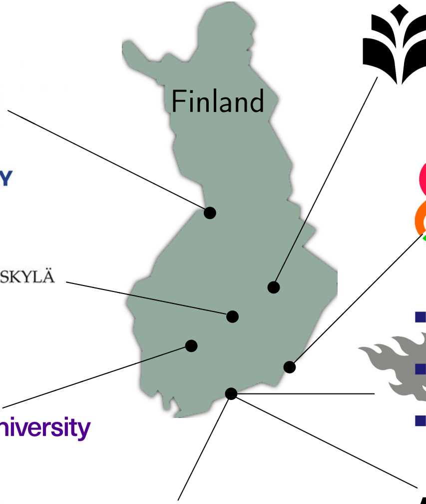
CENTRES OF EXCELLENCE IN RESEARCH



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Outline

1. Geometrization of elasticity
   - Newton's gravitation
   - Einstein's gravitation
   - Phonons
   - Elastic manifolds
2. Slowness surfaces
3. What about shear waves?
4. Results

Newton’s gravitation

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- The gravitational force exerted by the Sun causes the Earth's trajectory to curve.
- The force is described by a simple formula and the equation of motion is an ODE in $\mathbb{R}^3$.
- The Newtonian approach is straightforward to use and often a good model.

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- There is a relatively simple equation of motion for the planet: The geodesic equation is a non-linear ODE.
- There is a complicated equation of motion for the geometry itself: Einstein's field equation is a non-linear system of coupled PDEs.
- This model is harder to use but can reach phenomena inaccessible to Newtonian gravity and provides a more geometric way to see the essential structures.

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  - Newer view: The phonon goes straight in a curved geometry (along a geodesic), and the geometry is curved by variations in wave speed.

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- A norm on the tangent spaces defines a concept of distance.

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- The cosphere of the elastic geometry is the (qP) slowness surface.

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- In elastic geometry the distance between x and y is the shortest amount of time it takes for a wave to travel from x to y. (qP is fastest!)
- Distance is measured in units of time.
- Fermat's principle: Phonons go straight in this geometry.

Outline

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- Using Newton's second law with a restoring force given by Hooke's law leads to the elastic wave equation (EWE)

$$\sum_{j,k,l} \partial_j [c_{ijkl}(x) \partial_k u_l(x,t)] - \rho(x) \partial_t^2 u_i(x,t) = 0,$$

where $u(x,t)$ is a small displacement field.

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We are not looking at the full solution $u(x, t)$ of the EWE but only its singularities.

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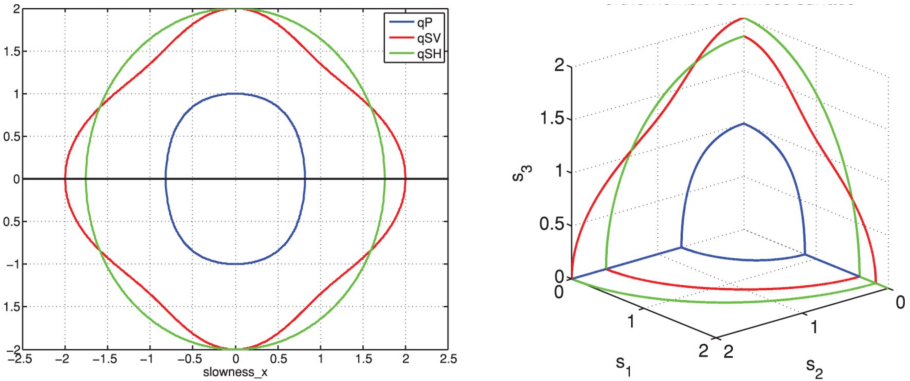
- These set of possible ps is the *slowness surface* at x.



The slowness surface. Smaller slowness $\iff$ faster wave.
The fastest one (quasi-pressure) is best behaved.

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- We call it the *slowness polynomial*.

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Much of differential geometry breaks apart!

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  - A simple description of the cosphere (the slowness surface): the slowness polynomial.

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- The slowness surface is an algebraic variety.
- Algebraic geometry studies these “manifolds” — a new box of tools.
- The different branches of the slowness surface appear independent, but they come from the same polynomial.
- Question: What does the qP branch tell about qS?

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- Answer: In the Zariski closure of the set.

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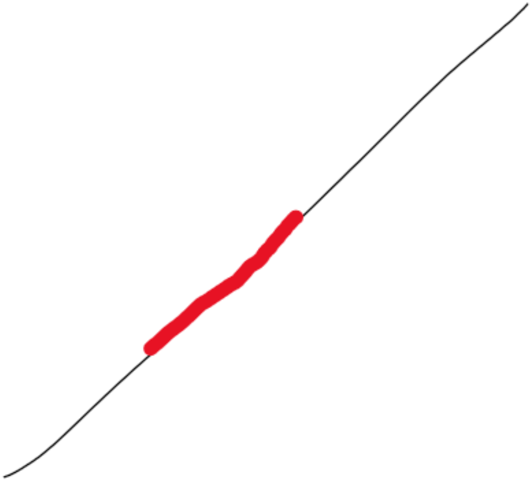

If a polynomial vanishes on the red set, it has to vanish in the black set. The black set is the Zariski closure of the red set.

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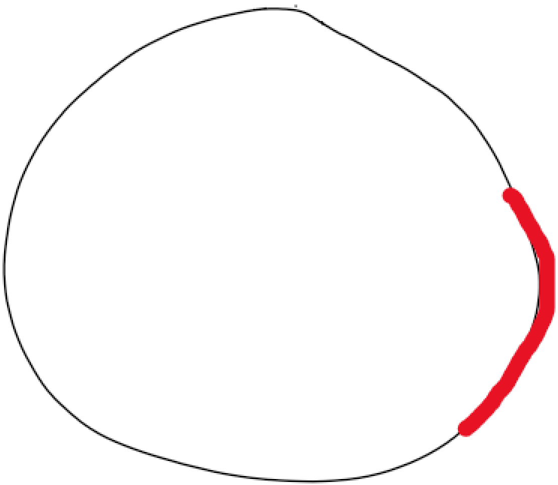

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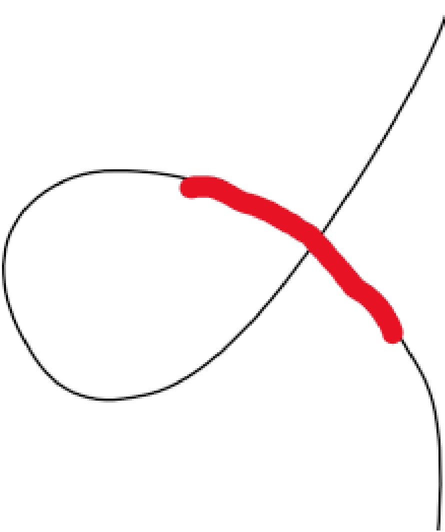

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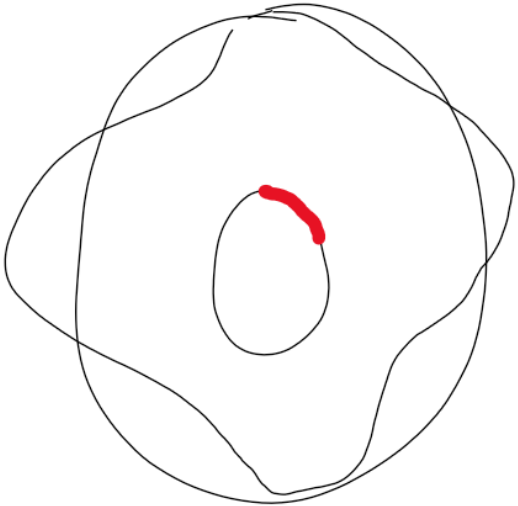

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Anisotropy helps!

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- A more geometric formulation: Given some boundary data, find the cosphere (slowness surface) at every point.
- Practical goal for a geometer: Given some boundary data, find the Finsler manifold.
- It is enough to study the geometry of the best-behaved qP waves, and algebra (often) gives qS for free.

Dix

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Measurements of travel times from interior earthquakes at the surface determine the qP slowness surfaces everywhere.

Scattering relation

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Single scattering measurements with directions of in-going and out-going waves determine the qP slowness surfaces everywhere.

Thank you!

Key ideas:

- Geometrization turns elastic waves into Finsler geodesics.
- Geometric inverse problems are easier to solve.
- Algebraic geometry relates qP to qS.

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