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PARTIAL DATA PROBLEMS IN SCALAR AND VECTOR FIELD TOMOGRAPHY

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ABSTRACT. We prove that if $P(D)$ is some constant coefficient partial differential operator and f is a scalar field such that $P(D)f$ vanishes in a given open set, then the integrals of f over all lines intersecting that open set determine the scalar field uniquely everywhere. This is done by proving a unique continuation property of fractional Laplacians which implies uniqueness for the partial data problem. We also apply our results to partial data problems of vector fields.

1. Introduction

Let f be a scalar field and $V \subset \mathbb{R}^n$ a nonempty open set where $n \geq 2$. We study the following partial data problem in X-ray tomography: can we say something about f if we know the integrals of f over all lines intersecting V? Especially, we are interested in the uniqueness problem which can be formulated in terms of the X-ray transform X_0 as follows: if $X_0 f = 0$ on all lines which intersect V, does it follow that $f = 0$? In general, the answer is no [29] and one has to put some conditions on $f|_V$. We prove that if there is some constant coefficient partial differential operator $P(D)$ such that $P(D)f|_V = 0$ and $X_0 f = 0$ on all lines intersecting V, then $f = 0$. This generalizes a recent partial data result in [20]. As a special case we obtain that if f is for example polynomial or (poly)harmonic in V, then f is uniquely determined by its partial X-ray data.

The partial data result is proved by using a unique continuation property of fractional Laplacian $(-\Delta)^s$. We prove that if $s \in (-n/2, \infty) \setminus \mathbb{Z}$ and there is some constant coefficient partial differential operator $P(D)$ such that $P(D)f|_V = (-\Delta)^s f|_V = 0$, then $f = 0$. This generalizes earlier results about unique continuation of fractional Laplacians [6, 14]. The unique continuation of $(-\Delta)^s$ implies a unique continuation result for the normal operator N_0 of the X-ray transform X_0, and the uniqueness for the partial data problem follows directly from the unique continuation of N_0. This approach which uses the unique continuation of the normal operator in proving uniqueness for partial data problems was also used in [20, 21].

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We also study partial data problems of vector fields. Let F be a vector field and denote by $\mathrm{d}F$ its exterior derivative or curl which components are $(\mathrm{d}F)_{ij} = \partial_i F_j - \partial_j F_i$. We prove that if there are some constant coefficient partial differential operators $P_{ij}(D)$ such that $P_{ij}(D)(\mathrm{d}F)_{ij}|_V = 0$ and the integrals of F over all lines intersecting V vanish, then F must be a potential field (F is the gradient of some scalar field). This is a generalization of a recent result in [21]. The partial data result is proved by using a relation between the normal operator of the X-ray transform of scalar fields and the normal operator of the X-ray transform of vector fields (see lemma 4.4). This allows one to reduce the partial data problem for the vector field F to partial data problems for the scalar fields $(\mathrm{d}F)_{ij}$. As a special case we obtain that if F is for example componentwise polynomial or (poly)harmonic in V, then the solenoidal part of F is uniquely determined by the partial X-ray data of F.

The partial data problems we study have a relation to the region of interest (ROI) tomography [4, 23, 24, 29, 46]. The main goal in such imaging problems is to determine the attenuation inside a small part of a human body (region of interest) by using only the X-ray data on lines which go through the ROI. This for example reduces the needed X-ray dose which is given to the patient. Our results imply that if the attenuation f satisfies $P(D)f|_V = 0$ for some open subset V of the ROI and some constant coefficient partial differential operator $P(D)$, then f is uniquely determined by its partial X-ray data on lines which intersect the ROI. Note that f is uniquely determined not only in the ROI but also outside the ROI. This holds for example if the attenuation is polynomial or (poly)harmonic in a small subregion of the ROI. In general, f does not have to be smooth and it can have singularities in the ROI. We also note that our proof for uniqueness does not give stability for the partial data problem. Especially, outside the ROI we have invisible singularities which cannot be seen by the X-ray data and the reconstruction of such singularities is not stable (see remark 1.5 and [25, 34, 35]).

Similar ROI tomography problems can be studied in the case of vector fields. In vector field tomography the usual objective is to determine the velocity field of a fluid flow using acoustic travel time or Doppler backscattering measurements [30, 31, 39]. Assuming that the velocity of the fluid flow is much smaller than the speed of the propagating signal one can linearize the problem. Linearization then leads to the X-ray transform of the velocity field. Our results imply that if the velocity field F satisfies $P_{ij}(D)(\mathrm{d}F)_{ij}|_V = 0$ for some open subset V of the ROI and some constant coefficient partial differential operators $P_{ij}(D)$, then the solenoidal part of F is uniquely determined everywhere by the partial X-ray data of F on lines intersecting the ROI. Examples of such velocity fields are those which are componentwise polynomial or (poly)harmonic in a small subregion of the ROI. As in the scalar case, F can have singularities in the ROI, and our proof does not give stability for the partial data problem (since it is based on reduction to the scalar case).

The article is organized as follows. In section 1.1 we introduce our notation, in section 1.2 we give our main theorems and in section 1.3 we discuss some related results. We go through the theory of distributions and the X-ray transform in section 2, and study the space of admissible functions in section 3. Finally, we prove our main results in section 4.

1.1. **Notation.** We quickly go through the notation used in our main theorems. More detailed information about distributions and the X-ray transform of scalar and vector fields can be found in section 2.

We denote by f a scalar field. The set $\mathscr{O}_C'(\mathbb{R}^n)$ is the space of rapidly decreasing distributions and the space $\mathcal{E}'(\mathbb{R}^n) \subset \mathscr{O}_C'(\mathbb{R}^n)$ consists of compactly supported distributions. The subset $C_\infty(\mathbb{R}^n) \subset \mathscr{O}_C'(\mathbb{R}^n)$ is the set of all continuous functions which decay faster than any polynomial at infinity. We let X_0 be the X-ray transform of scalar fields and it maps a function to its line integrals. The normal operator is $N_0 = X_0^* X_0$ where X_0^* is the adjoint of X_0.

We let $H^r(\mathbb{R}^n)$ be the fractional L^2-Sobolev space of order $r \in \mathbb{R}$ and $H^{-\infty}(\mathbb{R}^n) = \bigcup_{r\in\mathbb{R}} H^r(\mathbb{R}^n)$. We define the fractional Laplacian as $(-\Delta)^s f = \mathcal{F}^{-1}(|\cdot|^{2s} \hat{f})$ where $\hat{f} = \mathcal{F}(f)$ is the Fourier transform of f and $\mathcal{F}^{-1}$ is the inverse Fourier transform. The fractional Laplacian $(-\Delta)^s$ is well-defined in $\mathscr{O}_C'(\mathbb{R}^n)$ for all $s \in (-n/2, \infty) \setminus \mathbb{Z}$ and in $H^r(\mathbb{R}^n)$ for all $s \in (-n/4, \infty) \setminus \mathbb{Z}$.

We denote by $\mathcal{P}$ the set of all polynomials in $\mathbb{R}^n$ with complex coefficients with the convention that the zero polynomial $P \equiv 0$ does not belong to $\mathcal{P}$. A polynomial $P \in \mathcal{P}$ of degree $m \in \mathbb{N}$ induces a constant coefficient partial differential operator $P(D)$ of order $m \in \mathbb{N}$ by setting $P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ where $a_\alpha \in \mathbb{C}$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = -i\partial_j$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multi-index such that $|\alpha| = \alpha_1 + \ldots + \alpha_n$. The set of admissible functions $\mathcal{A}_V$ is defined as

$$\mathcal{A}_V = \{f \in H^{-\infty}(\mathbb{R}^n) : P(D)f|_V = 0 \text{ for some } P \in \mathcal{P}\} \tag{1}$$

where $V \subset \mathbb{R}^n$ is some nonempty open set.

We denote by F a vector field. The notation $F \in (\mathcal{E}'(\mathbb{R}^n))^n$ means that $F = (F_1, \ldots, F_n)$ where $F_i \in \mathcal{E}'(\mathbb{R}^n)$ for all $i = 1, \ldots, n$. The exterior derivative of F is written in components as $(\mathrm{d}F)_{ij} = \partial_i F_j - \partial_j F_i$. For scalar fields ϕ the notation $\mathrm{d}\phi$ denotes the gradient of ϕ. We let X_1 be the X-ray transform of vector fields which maps a vector field to its line integrals. The normal operator is $N_1 = X_1^* X_1$ where X_1^* is the adjoint of X_1.

1.2. **Main results.** In this section we give our main theorems. The proofs of the results can be found in section 4.

Our main theorem is the following unique continuation result for the fractional Laplacian.

Theorem 1.1. *Let $n \geq 1$, $s \in (-n/4, \infty) \setminus \mathbb{Z}$ and $f \in \mathcal{A}_V$ where $V \subset \mathbb{R}^n$ is some nonempty open set. If $(-\Delta)^s f|_V = 0$, then $f = 0$. If $f \in \mathscr{O}_C'(\mathbb{R}^n) \cap \mathcal{A}_V$, then the claim holds for $s \in (-n/2, \infty) \setminus \mathbb{Z}$.*

Theorem 1.1 generalizes the result in [6] (see lemma 4.1) where one assumes that $(-\Delta)^s f|_V = f|_V = 0$. In fact, theorem 1.1 is proved by reducing the claim to the case treated in [6, Theorem 1.1] (see section 4). The meaning of the condition $f \in \mathcal{A}_V$ is discussed in section 3 (see remark 3.3). When $s \in (-n/2, -n/4] \setminus \mathbb{Z}$, we need to have $f \in \mathscr{O}_C'(\mathbb{R}^n)$ so that $(-\Delta)^s f$ is well-defined and we can use lemma 4.1 in the proof of theorem 1.1.

Remark 1.2. *If $f \in \mathcal{E}'(\mathbb{R}^n)$, then instead of assuming $(-\Delta)^s f|_V = 0$ in theorem 1.1 we could only require that $(-\Delta)^s f$ vanishes to infinite order at some point $x_0 \in V$, i.e. $\partial^\beta((-\Delta)^s f)(x_0) = 0$ for all $\beta \in \mathbb{N}^n$. This follows since a corresponding unique continuation result is known for $f \in \mathcal{E}'(\mathbb{R}^n)$ under the assumptions $f|_V = 0$ and $\partial^\beta((-\Delta)^s f)(x_0) = 0$ for all $\beta \in \mathbb{N}^n$ (see corollary 4 on page 12 in [6]), and constant coefficient partial differential operators $P(D)$ commute with fractional Laplacians and ordinary derivatives. Therefore we can use the same proof to prove this slightly stronger result (see the proof of theorem 1.1).*

From the unique continuation of fractional Laplacians we immediately obtain the following unique continuation result for the normal operator of the X-ray transform of scalar fields. The reason is that the normal operator can be written as $N_0 = (-\Delta)^{-1/2}$ up to a constant factor (see section 2.2).

Theorem 1.3. *Let $n \geq 2$ and $f \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ or $f \in C_\infty(\mathbb{R}^n) \cap \mathcal{A}_V$ where $V \subset \mathbb{R}^n$ is some nonempty open set. If $N_0 f|_V = 0$, then $f = 0$.*

Theorem 1.3 is a generalization of the result in [20] where one assumes $N_0 f|_V = f|_V = 0$. When $f \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$, we could replace the assumption $N_0 f|_V = 0$ with the requirement that $N_0 f$ vanishes to infinite order at some point $x_0 \in V$ (see remark 1.2). In order to use theorem 1.1 in the case $s = -1/2$ and $n \geq 2$, and to guarantee that $N_0 f$ is well-defined, we need to have $f \in \mathcal{E}'(\mathbb{R}^n) \subset \mathscr{O}_C'(\mathbb{R}^n)$ or $f \in C_\infty(\mathbb{R}^n) \subset \mathscr{O}_C'(\mathbb{R}^n)$ in theorem 1.3.

The unique continuation of N_0 implies uniqueness for the following partial data problem.

Theorem 1.4. *Let $n \geq 2$ and $f \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ or $f \in C_\infty(\mathbb{R}^n) \cap \mathcal{A}_V$ where $V \subset \mathbb{R}^n$ is some nonempty open set. If $X_0 f = 0$ on all lines intersecting V, then $f = 0$.*

Theorem 1.4 generalizes theorem 1.2 in [20], where one assumes $f|_V = 0$, to the case $P(D)f|_V = 0$ for some $P \in \mathcal{P}$. We note that if f is polynomial in V, then there is $P \in \mathcal{P}$ such that $P(D)f|_V = 0$. Hence those scalar fields which are polynomial in V can be uniquely determined from their X-ray data on lines intersecting V. This special case of theorem 1.4 is previously known in two dimensions [23, 46]. We also note that theorem 1.4 includes much larger class of functions than just polynomials. The scalar field f can be (poly)harmonic in V and f can also have singularities in V if f is for example a non-smooth solution to the wave equation (see section 3 for more examples of admissible functions).

It is important to notice that from the vector space structure of admissible functions $\mathcal{A}_V$ it follows that theorem 1.4 is indeed a uniqueness result: if f_1 and f_2 satisfy $P_1(D)f_1|_V = P_2(D)f_2|_V = 0$ for some $P_1, P_2 \in \mathcal{P}$ and $X_0 f_1 = X_0 f_2$ on all lines intersecting V, then $f_1 = f_2$ (see proposition 3.4 and remark 3.5 for more details). Especially, the equality of the X-ray data on all lines intersecting V implies that the scalar fields are equal everywhere even though f_1 and f_2 a priori can have very different behaviour in V since $P_1(D)$ can be different from $P_2(D)$.

Remark 1.5. *We note that our proof for theorem 1.4 gives only uniqueness but not stability for the partial data problem. In theorem 1.4 we eventually* *have to assume that f is not supported in V since otherwise we would have $P(D)f = 0$ everywhere and therefore $f = 0$ without assuming anything about the X-ray data (see the proof of theorem 1.1). When f is supported outside V we do not have access to all singularities of f via the X-ray data, i.e. we have invisible singularities outside V. It is known that the recovery of such invisible singularities is not stable* [25, 34, 35].

Remark 1.6. *We note that similar results as in theorems 1.3 and 1.4 also hold for the d-plane transform when d is odd (see corollaries 1 and 2 on page 6 in* [6]*). The d-plane transform takes a scalar field and integrates it over d-dimensional affine planes where $0 < d < n$. The case $d = 1$ corresponds to the X-ray transform. The normal operator N_d of the d-plane transform can be expressed as $N_d = (-\Delta)^{-d/2}$ up to a constant factor (see* [6, 16]*). Hence N_d admits the same unique continuation property as in theorem 1.1 for functions in $\mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ or $C_\infty(\mathbb{R}^n) \cap \mathcal{A}_V$ provided d is odd. The unique continuation of N_d then implies a similar uniqueness result as in theorem 1.4 for a partial data problem of the d-plane transform when d is odd.*

From the unique continuation of fractional Laplacians we also obtain a partial data result for the X-ray transform of vector fields. The normal operators satisfy the relationship $\mathrm{d}(N_1 F) = N_0(\mathrm{d}F)$ up to a constant factor (see lemma 4.4). Hence the unique continuation and partial data problems of vector fields can be reduced to the corresponding problems for scalar fields, namely the components $(\mathrm{d}F)_{ij}$.

The next theorem generalizes the result in [21] where the authors assume that $\mathrm{d}F|_V = 0$ instead of $(\mathrm{d}F)_{ij} \in \mathcal{A}_V$.

Theorem 1.7. *Let $n \geq 2$ and $F \in (\mathcal{E}'(\mathbb{R}^n))^n$ such that $(\mathrm{d}F)_{ij} \in \mathcal{A}_V$ for all $i, j = 1, \ldots, n$ where $V \subset \mathbb{R}^n$ is some nonempty open set. If $X_1 F = 0$ on all lines intersecting V, then $\mathrm{d}F = 0$. Especially, $F = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$.*

The conclusion $F = \mathrm{d}\phi$ in theorem 1.7 is equivalent to that the solenoidal part F^{s} vanishes in the solenoidal decomposition $F = F^{\mathrm{s}} + \mathrm{d}\phi$ (see e.g. [41]). Therefore theorem 1.7 can be seen as a solenoidal injectivity result in terms of partial data (see [21] and [33, 41]). Theorem 1.7 holds also for vector fields $F \in (\mathscr{S}(\mathbb{R}^n))^n$ which components are Schwartz functions since in that case $(\mathrm{d}F)_{ij} \in C_\infty(\mathbb{R}^n) \cap \mathcal{A}_V$.

We note that if the components F_i are all polynomial in V, then also $(\mathrm{d}F)_{ij}$ are all polynomial in V. Hence there are some $P_{ij} \in \mathcal{P}$ such that $P_{ij}(D)(\mathrm{d}F)_{ij}|_V = 0$ and therefore $(\mathrm{d}F)_{ij} \in \mathcal{A}_V$. This means that solenoidal vector fields which are polynomial in V can be uniquely determined from their X-ray data on lines intersecting V. However, this is only a small subset of admissible vector fields: F can be for example componentwise (poly)harmonic in V and more generally F can also have singularities in V.

Remark 1.8. *Theorems 1.1 and 1.3 imply a unique continuation result for N_1: if $F \in (\mathcal{E}'(\mathbb{R}^n))^n$ satisfies $(\mathrm{d}F)_{ij} \in \mathcal{A}_V$ for all $i, j = 1, \ldots, n$ and $N_1 F|_V = 0$, then $\mathrm{d}F = 0$. This follows since $\mathrm{d}(N_1 F) = N_0(\mathrm{d}F)$ up to a constant factor (see lemma 4.4) and one can use theorem 1.3 for the components $(\mathrm{d}F)_{ij} \in \mathcal{E}'(\mathbb{R}^n)$. One also obtains a stronger version where one can replace the assumption $N_1 F|_V = 0$ with the requirement that $\mathrm{d}(N_1 F)$ vanishes componentwise to infinite order at some point $x_0 \in V$ (see remark 1.2).*

1.3. **Related results.** There are some earlier unique continuation and partial data results for scalar and vector fields. The partial data problem for scalar fields has a unique solution if $f|_V$ vanishes [4, 20, 24], $f|_V$ is polynomial or piecewise polynomial [23, 24, 46] or $f|_V$ is real analytic [23]. A complementary result is the Helgason support theorem: if the integrals of f vanish on all lines not intersecting a given compact and convex set, then f has to vanish outside that set [16, 42]. The normal operator of the X-ray transform of scalar fields has a unique continuation property under the assumptions $N_0 f|_V = f|_V = 0$ [20]. This is a special case of a more general unique continuation property of fractional Laplacians [6, 14]. There are also partial data and unique continuation results for the d-plane transform of scalar fields when d is odd, including the X-ray transform as a special case $d = 1$ (see [6] and remark 1.6).

The partial data problem of vector fields is known to be uniquely solvable up to potential fields, if $\mathrm{d}F|_V = 0$ [21]. Similarly, the normal operator of the X-ray transform of vector fields has a unique continuation property under the assumptions $N_1 F|_V = \mathrm{d}F|_V = 0$ [21]. There are other partial data results for vector fields where one knows the integrals of F over lines which are parallel to a finite set of planes [22, 38, 40] or which intersect a certain type of curve [9, 36, 44]. There is also a Helgason-type support theorem for vector fields: if the integrals of F vanish on all lines not intersecting a given compact and convex set, then $\mathrm{d}F$ vanishes outside that set [21, 42].

The normal operator of scalar fields, the normal operator of vector fields and the fractional Laplacian all admit stronger versions of the unique continuation property (see [6, 11, 12, 13, 20, 21, 37, 47] and remarks 1.2 and 1.8). Other applications of unique continuation of fractional Laplacians include fractional inverse problems. Especially, the unique continuation of $(-\Delta)^s$ is used to prove uniqueness for different versions of the fractional Calderón problem (see e.g. [1, 2, 5, 6, 7, 14]).

2. The X-ray transform and distributions

In this section we define the X-ray transform of scalar and vector fields, and introduce the distribution spaces we use in our main theorems. The basic theory of distributions and Sobolev spaces can be found in [15, 17, 27, 28, 43] and the X-ray transform is treated for example in [29, 41, 42].

2.1. **Distributions and Sobolev spaces.** We let $\mathcal{E}(\mathbb{R}^n)$ be the space of smooth functions, $\mathscr{S}(\mathbb{R}^n)$ is the Schwartz space and $\mathcal{D}(\mathbb{R}^n)$ is the space of compactly supported smooth functions. We equip all these spaces with their standard topologies. The corresponding duals are denoted by $\mathcal{E}'(\mathbb{R}^n)$, $\mathscr{S}'(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$. Elements in $\mathcal{E}'(\mathbb{R}^n)$ are identified with distributions of compact support and elements in $\mathscr{S}'(\mathbb{R}^n)$ are called tempered distributions.

We define the space of rapidly decreasing distributions $\mathscr{O}'_C(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ as follows: $f \in \mathscr{O}'_C(\mathbb{R}^n)$ if and only if $\hat{f} \in \mathscr{O}_M(\mathbb{R}^n)$ where $\hat{f} = \mathcal{F}(f)$ is the Fourier transform of tempered distributions. Here $\mathscr{O}_M(\mathbb{R}^n)$ is the space of polynomially growing smooth functions, i.e. $f \in \mathscr{O}_M(\mathbb{R}^n)$ if f and all its derivatives are polynomially bounded. We note that the Fourier transform is an isomorphism $\mathcal{F}\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ and also extends to an isomorphism $\mathcal{F}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. We have the inclusions $\mathcal{E}'(\mathbb{R}^n) \subset \mathscr{O}'_C(\mathbb{R}^n) \subset$

$\mathscr{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$. As a special case we have $\mathscr{S}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n) \subset \mathscr{O}'_C(\mathbb{R}^n)$ where $C_\infty(\mathbb{R}^n)$ is the set of all continuous functions which decay faster than any polynomial at infinity.

The fractional L^2-Sobolev space of order $r \in \mathbb{R}$ is defined as

$$H^r(\mathbb{R}^n) = \{f \in \mathscr{S}'(\mathbb{R}^n) : \langle\cdot\rangle^r \hat{f} \in L^2(\mathbb{R}^n)\} \tag{2}$$

where $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. The space $H^r(\mathbb{R}^n)$ is equipped with the norm

$$\|f\|_{H^r(\mathbb{R}^n)} = \left\|\langle\cdot\rangle^r \hat{f}\right\|_{L^2(\mathbb{R}^n)} \tag{3}$$

and $H^r(\mathbb{R}^n)$ becomes a separable Hilbert space for every $r \in \mathbb{R}$. It follows that the spaces are nested, i.e. $H^r(\mathbb{R}^n) \hookrightarrow H^t(\mathbb{R}^n)$ continuously when $r \geq t$. One can isomorphically identify $H^{-r}(\mathbb{R}^n)$ with the dual $(H^r(\mathbb{R}^n))^*$ for all $r \in \mathbb{R}$. We define the following spaces

$$H^\infty(\mathbb{R}^n) = \bigcap_{r\in\mathbb{R}} H^r(\mathbb{R}^n), \quad H^{-\infty}(\mathbb{R}^n) = \bigcup_{r\in\mathbb{R}} H^r(\mathbb{R}^n). \tag{4}$$

It holds that $\mathscr{O}'_C(\mathbb{R}^n) \subset H^{-\infty}(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ and $\mathscr{S}(\mathbb{R}^n) \subset H^\infty(\mathbb{R}^n)$. Further, using the Sobolev embedding one can see that $H^\infty(\mathbb{R}^n) = C^\infty_{L^2}(\mathbb{R}^n)$ where $f \in C^\infty_{L^2}(\mathbb{R}^n)$ if f is smooth and f and all its derivatives belong to $L^2(\mathbb{R}^n)$ (see [15, Theorem 6.12]).

The fractional Laplacian is defined as

$$(-\Delta)^s f = \mathcal{F}^{-1}(|\cdot|^{2s} \hat{f}) \tag{5}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform of tempered distributions. It follows that $(-\Delta)^s f$ is well-defined as a tempered distribution for $f \in \mathscr{O}'_C(\mathbb{R}^n)$ when $s \in (-n/2, \infty) \setminus \mathbb{Z}$, and for $f \in H^r(\mathbb{R}^n)$ when $s \in (-n/4, \infty) \setminus \mathbb{Z}$ (see [6, Section 2.2]). We have that $(-\Delta)^s \colon H^r(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)$ is continuous whenever $s \in (0, \infty) \setminus \mathbb{Z}$ and $(-\Delta)^s$ also admits a Poincaré-type inequality for $s \in (0, \infty) \setminus \mathbb{Z}$ (see [6]). We note that $(-\Delta)^s$ is a non-local operator in contrast to the ordinary Laplacian $(-\Delta)$. The non-locality implies a unique continuation property (see theorem 1.1 and lemma 4.1) which cannot hold for local operators.

We also use local versions of distributions and fractional Sobolev spaces. Let $\Omega \subset \mathbb{R}^n$ be an open set and $r \in \mathbb{R}$. We denote by $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$ etc. the test function and distribution spaces defined in Ω. We define the local Sobolev space $H^r(\Omega)$ as

$$H^r(\Omega) = \{g \in \mathcal{D}'(\Omega) : g = f|_\Omega \text{ for some } f \in H^r(\mathbb{R}^n)\}. \tag{6}$$

In other words, the space $H^r(\Omega)$ consists of restrictions of distributions $f \in H^r(\mathbb{R}^n)$. The local Sobolev space is equipped with the quotient norm

$$\|g\|_{H^r(\Omega)} = \inf\{\|f\|_{H^r(\mathbb{R}^n)} : f \in H^r(\mathbb{R}^n) \text{ such that } f|_\Omega = g\}. \tag{7}$$

Then $H^r(\Omega)$ becomes a separable Hilbert space and the restriction map $|_\Omega \colon H^r(\mathbb{R}^n) \to H^r(\Omega)$ is continuous. If $r \geq t$, then $H^r(\Omega) \hookrightarrow H^t(\Omega)$ continuously. One can also isomorphically identify $H^{-r}(\Omega)$ as the dual $(\widetilde{H}^r(\Omega))^*$ for every $r \in \mathbb{R}$ where $\widetilde{H}^r(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^r(\mathbb{R}^n)$ (see [3] and [27]). If $r \geq 0$, then $H^r(\Omega) \subset W^r(\Omega)$ where $W^r(\Omega)$ is the Sobolev-Slobodeckij space which is defined by using weak derivatives of L^2-functions (see [27] for a precise definition). If Ω is a Lipschitz domain, then we have the equality $H^r(\Omega) = W^r(\Omega)$ for all $r \geq 0$.

More generally, we define the vector-valued test function space $(\mathcal{D}(\mathbb{R}^n))^n$ by saying that $\varphi \in (\mathcal{D}(\mathbb{R}^n))^n$ if and only if $\varphi = (\varphi_1, \dots, \varphi_n)$ and $\varphi_i \in \mathcal{D}(\mathbb{R}^n)$ for all $i = 1, \dots, n$. A sequence converges to zero in $(\mathcal{D}(\mathbb{R}^n))^n$ if and only if all its components converge to zero in $\mathcal{D}(\mathbb{R}^n)$. We then define the space of vector-valued distributions $(\mathcal{D}'(\mathbb{R}^n))^n$ by saying that $F \in (\mathcal{D}'(\mathbb{R}^n))^n$ if and only if $F = (F_1, \dots, F_n)$ where $F_i \in \mathcal{D}'(\mathbb{R}^n)$ for all $i = 1, \dots, n$. The duality pairing is defined as $\langle F, \varphi \rangle = \sum_{i=1}^n \langle F_i, \varphi_i \rangle$. The test function spaces $(\mathcal{E}(\mathbb{R}^n))^n$ and $(\mathscr{S}(\mathbb{R}^n))^n$, and the corresponding distribution spaces $(\mathcal{E}'(\mathbb{R}^n))^n$ and $(\mathscr{S}'(\mathbb{R}^n))^n$ are defined analogously. The elements in $(\mathcal{E}'(\mathbb{R}^n))^n$ are called compactly supported vector-valued distributions. Vector-valued distributions are a special case of currents (continuous linear functionals in the space of differential forms, see [8, Section III]).

For $F \in (\mathcal{D}'(\mathbb{R}^n))^n$ we define the exterior derivative or curl of F as a matrix which components are $(\mathrm{d}F)_{ij} = \partial_i F_j - \partial_j F_i$. It follows from the Poincaré lemma (see e.g. [26, Theorem 2.1] and lemma 4.2) that if $\mathrm{d}F = 0$, then $F = \mathrm{d}\phi$ for some $\phi \in \mathcal{D}'(\mathbb{R}^n)$ where $\mathrm{d}\phi$ is the distributional gradient of ϕ.

2.2. **The X-ray transform of scalar fields.** Let $f \in \mathcal{D}(\mathbb{R}^n)$ be a scalar field. The X-ray transform X_0 is defined as

$$X_0 f(\gamma) = \int_\gamma f \mathrm{d}s \tag{8}$$

where γ is an oriented line in $\mathbb{R}^n$. When we parameterize the set of all oriented lines with the set

$$\Gamma = \{(z, \theta) : \theta \in S^{n-1},\ z \in \theta^\perp\} \tag{9}$$

the X-ray transform becomes

$$X_0 f(z, \theta) = \int_{\mathbb{R}} f(z + s\theta) \mathrm{d}s. \tag{10}$$

The adjoint or back-projection X_0^* is defined as

$$X_0^* \psi(x) = \int_{S^{n-1}} \psi(x - (x \cdot \theta)\theta, \theta) \mathrm{d}\theta \tag{11}$$

where $\psi \in \mathcal{E}(\Gamma)$. One then sees that $X_0 \colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\Gamma)$ and $X_0^* \colon \mathcal{E}(\Gamma) \to \mathcal{E}(\mathbb{R}^n)$ are continuous maps. Using duality we can define $X_0 \colon \mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(\Gamma)$ and $X_0^* \colon \mathcal{D}'(\Gamma) \to \mathcal{D}'(\mathbb{R}^n)$ as

$$\langle X_0 f, \varphi \rangle = \langle f, X_0^* \varphi \rangle \tag{12}$$

$$\langle X_0^* \psi, \eta \rangle = \langle \psi, X_0 \eta \rangle \tag{13}$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing.

The normal operator is $N_0 = X_0^* X_0$ and it can be expressed as the convolution

$$N_0 f(x) = 2(f * |\cdot|^{1-n})(x). \tag{14}$$

Using duality the normal operator extends to a map $N_0 \colon \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ and the convolution formula holds in the sense of distributions. The normal operator can be seen as the fractional Laplacian $(-\Delta)^{-1/2}$ up to a constant factor and we have the reconstruction formula

$$f = c_{0,n}(-\Delta)^{1/2} N_0 f \tag{15}$$

where $c_{0,n}$ is a constant which depends on dimension. Both X_0 and N_0 are also defined for functions $f \in C_\infty(\mathbb{R}^n)$.

2.3. **The X-ray transform of vector fields.** Let $F \in (\mathcal{D}(\mathbb{R}^n))^n$ be a vector field. The X-ray transform X_1 is defined as

$$X_1 F(\gamma) = \int_\gamma F \cdot \mathrm{d}\overline{s} \tag{16}$$

where γ is an oriented line. Using the parametrization Γ for oriented lines (see equation (9)) we have

$$X_1 F(z, \theta) = \int_{\mathbb{R}} F(z + s\theta) \cdot \theta \mathrm{d}s. \tag{17}$$

We define the adjoint X_1^* as the vector-valued operator

$$(X_1^* \psi)_i(x) = \int_{S^{n-1}} \theta_i \psi(x - (x \cdot \theta)\theta, \theta) \mathrm{d}\theta \tag{18}$$

where $\psi \in \mathcal{E}(\Gamma)$ is a scalar field in the space of oriented lines. One sees that $X_1 \colon (\mathcal{D}(\mathbb{R}^n))^n \to \mathcal{D}(\Gamma)$ and $X_1^* \colon \mathcal{E}(\Gamma) \to (\mathcal{E}(\mathbb{R}^n))^n$ are continuous and by duality we can define $X_1 \colon (\mathcal{E}'(\mathbb{R}^n))^n \to \mathcal{E}'(\Gamma)$ and $X_1^* \colon \mathcal{D}'(\Gamma) \to (\mathcal{D}'(\mathbb{R}^n))^n$ by setting

$$\langle X_1 F, \varphi \rangle = \langle F, X_1^* \varphi \rangle \tag{19}$$

$$\langle X_1^* \psi, \eta \rangle = \langle \psi, X_1 \eta \rangle . \tag{20}$$

We define the normal operator as $N_1 = X_1^* X_1$ and it can be expressed in terms of convolution

$$(N_1 F)_i = \sum_{j=1}^n \frac{2 x_i x_j}{|x|^{n+1}} * F_j. \tag{21}$$

The normal operator extends to a map $N_1 \colon (\mathcal{E}'(\mathbb{R}^n))^n \to (\mathcal{D}'(\mathbb{R}^n))^n$ by duality and the convolution formula holds in the sense of distributions. One has the reconstruction formula for the solenoidal part F^{s} in the solenoidal decomposition $F = F^{\mathrm{s}} + \mathrm{d}\phi$ (see for example [41, 42])

$$F^{\mathrm{s}} = c_{1,n}(-\Delta)^{1/2} N_1 F \tag{22}$$

where $c_{1,n}$ is a constant depending on dimension and $(-\Delta)^{1/2}$ operates componentwise on $N_1 F$. Both X_1 and N_1 are also defined for vector fields $F \in (\mathscr{S}(\mathbb{R}^n))^n$.

3. Partial differential operators and admissible functions

In this section we introduce constant coefficient partial differential operators and also study the space of admissible functions $\mathcal{A}_V$ in more detail. A comprehensive treatment of constant coefficient partial differential operators can be found in Hörmander's book [18].

Let us denote by $\mathcal{P}$ the set of all polynomials in $\mathbb{R}^n$ with complex coefficients excluding the zero polynomial $P \equiv 0$. A polynomial $P \in \mathcal{P}$ of degree

$m \in \mathbb{N}$ can be identified with the constant coefficient partial differential operator $P(D)$ of order $m \in \mathbb{N}$ as

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, \quad a_\alpha \in \mathbb{C}, \tag{23}$$

where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = -i\partial_j$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index such that $|\alpha| = \alpha_1 + \ldots + \alpha_n$. In fact, using the Fourier transform one sees that

$$\widehat{P(D)} = P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha \tag{24}$$

where $\xi \in \mathbb{R}^n$ and $\xi^\alpha = \xi^{\alpha_1} \cdots \xi^{\alpha_n}$. The polynomial $P(\xi)$ is also known as the full symbol of $P(D)$. If $g \in \mathcal{D}'(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is an open set, then one can define the distributional derivative $P(D)g \in \mathcal{D}'(\Omega)$ by duality. Further, it holds that $P(D)\colon H^r(\Omega) \to H^{r-m}(\Omega)$ is continuous with respect to the quotient norm [28, Theorem 12.15] (see equation (7)).

The set of admissible functions $\mathcal{A}_V$ which we use in our main theorems can be written as the union

$$\mathcal{A}_V = \bigcup_{\substack{P \in \mathcal{P} \\ r \in \mathbb{R}}} \mathcal{H}^r_{P,V}(\mathbb{R}^n) = \bigcup_{\substack{P \in \mathcal{P} \\ r \in \mathbb{R}}} \{f \in H^r(\mathbb{R}^n) : P(D)f|_V = 0\} \tag{25}$$

where $V \subset \mathbb{R}^n$ is some nonempty open set and $\mathcal{H}^r_{P,V}(\mathbb{R}^n) = \{f \in H^r(\mathbb{R}^n) : P(D)f|_V = 0\}$. We note that $\mathcal{A}_V \subset H^{-\infty}(\mathbb{R}^n)$. The following proposition implies that the sets $\mathcal{H}^r_{P,V}(\mathbb{R}^n)$ in the union (25) are also Hilbert spaces.

Proposition 3.1. *The subset $\mathcal{H}^r_{P,V}(\mathbb{R}^n) \subset H^r(\mathbb{R}^n)$ is a separable Hilbert space for all $r \in \mathbb{R}$, $P \in \mathcal{P}$ and nonempty open set $V \subset \mathbb{R}^n$.*

Proof. Clearly $\mathcal{H}^r_{P,V}(\mathbb{R}^n)$ is a linear subspace of $H^r(\mathbb{R}^n)$. Let $f_k \in \mathcal{H}^r_{P,V}(\mathbb{R}^n)$ be a sequence such that $f_k \to f$ in $H^r(\mathbb{R}^n)$. Then by the continuity of the restriction map $|_V\colon H^r(\mathbb{R}^n) \to H^r(V)$ we have that $f_k|_V \to f|_V$ in $H^r(V)$. From the continuity of $P(D)\colon H^r(V) \to H^{r-m}(V)$ we obtain that $0 = P(D)f_k|_V \to P(D)f|_V$ in $H^{r-m}(V)$, implying that $f \in \mathcal{H}^r_{P,V}(\mathbb{R}^n)$. Therefore $\mathcal{H}^r_{P,V}(\mathbb{R}^n)$ is a closed subspace of the separable Hilbert space $H^r(\mathbb{R}^n)$ and hence itself a separable Hilbert space. □

Remark 3.2. *We note that in the smooth case we have that $\mathcal{E}_{P,V}(\mathbb{R}^n) = \{f \in \mathcal{E}(\mathbb{R}^n) : P(D)f|_V = 0\} \subset \mathcal{E}(\mathbb{R}^n)$ is a closed subspace of $\mathcal{E}(\mathbb{R}^n)$ and hence a Fréchet space. More generally, $\mathcal{D}'_{P,V}(\mathbb{R}^n) = \{f \in \mathcal{D}'(\mathbb{R}^n) : P(D)f|_V = 0\} \subset \mathcal{D}'(\mathbb{R}^n)$ is sequentially closed in $\mathcal{D}'(\mathbb{R}^n)$ under the weak* convergence. These two facts follow from the continuity of $P(D)\colon \mathcal{E}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)$ and $P(D)\colon \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ with respect to the standard topologies. More topological properties of kernels of constant coeffiecient partial differential operators can be found in* [45].

Remark 3.3. *The interpretation of the condition $f \in \mathcal{A}_V$ is the following. If $f \in \mathcal{A}_V$, then there is some $r \in \mathbb{R}$ and some $P \in \mathcal{P}$ such that $f \in H^r(\mathbb{R}^n)$ and $P(D)f|_V = 0$. The distributional derivatives commute with restrictions, i.e. $P(D)f|_V = P(D)(f|_V)$ where $f|_V \in \mathcal{D}'(V)$. Since $f \in H^r(\mathbb{R}^n)$ we see that $f|_V$ is not only a distribution but in addition $f|_V \in H^r(V)$ for some*

$r \in \mathbb{R}$. Therefore the existence of $r \in \mathbb{R}$ and $P \in \mathcal{P}$ for which $P(D)f|_V = 0$ means that $f|_V \in H^r(V)$ and $f|_V$ is a weak solution to some homogeneous constant coefficient partial differential equation. In other words, $f|_V$ satisfies

$$\sum_{|\alpha|\leq m} a_\alpha D^\alpha(f|_V) = 0, \quad f|_V \in \bigcup_{r\in\mathbb{R}} H^r(V), \tag{26}$$

for some coefficients $a_\alpha \in \mathbb{C}$ and some integer $m \in \mathbb{N}$.

The following proposition is important in the uniqueness of the partial data problem.

Proposition 3.4. *The set $\mathcal{A}_V \subset H^{-\infty}(\mathbb{R}^n)$ is a vector space for every nonempty open set $V \subset \mathbb{R}^n$.*

Proof. Let $f_1, f_2 \in \mathcal{A}_V$ and $\lambda \in \mathbb{C}$. This means that $f_1 \in H^{r_1}(\mathbb{R}^n)$, $f_2 \in H^{r_2}(\mathbb{R}^n)$ and $P_1(D)f_1|_V = P_2(D)f_2|_V = 0$ for some $r_1, r_2 \in \mathbb{R}$ and $P_1, P_2 \in \mathcal{P}$. It follows that $f_1 + \lambda f_2 \in H^r(\mathbb{R}^n)$ where $r = \min\{r_1, r_2\}$ since the spaces $H^t(\mathbb{R}^n)$, $t \in \mathbb{R}$, are nested vector spaces. We also have that $P_1(D)P_2(D)(f_1 + \lambda f_2)|_V = 0$ since the distributional derivatives commute $P_1(D)P_2(D) = P_2(D)P_1(D)$. This implies that $f_1 + \lambda f_2 \in \mathcal{A}_V$, i.e. $\mathcal{A}_V$ is a linear subspace of the vector space $H^{-\infty}(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$. □

Remark 3.5. *The vector space structure of $\mathcal{A}_V$ is important since it implies that the partial data results we have proved in this article are indeed uniqueness results. Namely, if $f_1, f_2 \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ (or $f_1, f_2 \in C_\infty(\mathbb{R}^n) \cap \mathcal{A}_V$) such that $X_0 f_1 = X_0 f_2$ on all lines intersecting V, then $f_1 - f_2 \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ (or $f_1 - f_2 \in C_\infty(\mathbb{R}^n) \cap \mathcal{A}_V$) and $X_0(f_1 - f_2) = 0$ on all lines intersecting V. Theorem 1.4 then implies that $f_1 - f_2 = 0$, i.e. the solution to the partial data problem is unique.*

We list some examples of admissible functions. We have that the function $f \in H^{-\infty}(\mathbb{R}^n)$ belongs to $\mathcal{A}_V$, if

- f is polyharmonic in V, i.e. $(-\Delta)^k f|_V = 0$ for some $k \in \mathbb{N}$.
- f is polynomial in V.
- f is independent of one of the variables $x_1, \dots, x_n$ in V.
- $f(x) = q(x)e^{ix\cdot\zeta}$ in V where $q(x)$ is a suitable polynomial and $\zeta \in \mathbb{C}^n$ is a generalized frequency. Especially, if f is of the form $f(x) = e^{ix\cdot\xi_0}$ in V where $\xi_0 \in \mathbb{C}^n$ is a zero of $P \in \mathcal{P}$, then $P(D)f|_V = 0$.

Further, it holds that for convex sets V and a fixed $P \in \mathcal{P}$ the linear span of solutions of the form $q(x)e^{ix\cdot\zeta}$ is dense in the space of all smooth solutions of $P(D)g = 0$ in V (see [17, Theorem 7.3.6] and a more general result [18, Theorem 10.5.1]).

We note that if $P(D)$ is a hypoelliptic operator, then the condition $P(D)f|_V = 0$ already implies that f is smooth in V (see [18, 28]). Basic examples of hypoelliptic operators are elliptic operators such as integer powers of Laplacians ($(-\Delta)^k$ where $k \in \mathbb{N}$) and also the non-elliptic heat operator $\partial_t - \Delta$. However, there are non-smooth distributions $f|_V$ which satisfy the condition $P(D)f|_V = 0$ for some $P \in \mathcal{P}$ and therefore f can have singularities in V. For example, the wave operator $\partial_t^2 - \Delta$ is not hypoelliptic and has non-smooth weak solutions.

4. Proofs of the main theorems

In this section we prove our main theorems. We need a few auxiliary results. The first one is a unique continuation result for fractional Laplacians and the second one is the Poincaré lemma for compactly supported vector-valued distributions.

Lemma 4.1 ([6, Theorem 1.1]). *Let $n \geq 1$, $s \in (-n/4, \infty) \setminus \mathbb{Z}$ and $u \in H^t(\mathbb{R}^n)$ where $t \in \mathbb{R}$. If $(-\Delta)^s u|_V = 0$ and $u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $u = 0$. The claim holds also for $s \in (-n/2, -n/4] \setminus \mathbb{Z}$ if $u \in \mathscr{O}'_C(\mathbb{R}^n)$.*

Lemma 4.2 (Poincaré lemma). *Let $U \in (\mathcal{E}'(\mathbb{R}^n))^n$ such that $\mathrm{d}U = 0$. Then there is $\phi \in \mathcal{E}'(\mathbb{R}^n)$ such that $U = \mathrm{d}\phi$.*

The proof of lemma 4.2 can be found for example in [19, 26]. The third lemma is a known result about the zero set of multivariate polynomials.

Lemma 4.3 ([32, Lemma on p.1]). *Let $Q = Q(x)$ be a non-zero multivariate polynomial of order $m \in \mathbb{N}$*

$$Q(x) = \sum_{|\alpha| \leq m} b_\alpha x^\alpha = \sum_{|\alpha| \leq m} b_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad b_\alpha \in \mathbb{C}, \tag{27}$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index such that $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Then the set $Z_Q = \{x \in \mathbb{R}^n : Q(x) = 0\}$ has Lebesgue measure zero.

Lemma 4.3 is proved in [32] for real coefficients but the result holds also for complex coefficients by splitting $b_\alpha \in \mathbb{C}$ to its real and imaginary parts. We note that the set Z_Q is Zariski closed but not the whole space $\mathbb{R}^n$. From the coarseness of the Zariski topology (i.e. there are relatively few closed sets) one can already deduce that the set Z_Q must be small in topological sense (see e.g. [10, Chapter 15.2]).

The next lemma shows how the normal operator of the X-ray transform of vector fields is related to the normal operator of scalar fields (see also [21, Proof of theorem 1.1]).

Lemma 4.4. *Let $F \in (\mathcal{E}'(\mathbb{R}^n))^n$. Then $\mathrm{d}(N_1 F) = (n-1)^{-1} N_0(\mathrm{d}F)$ holds componentwise where N_0 acts on the components $(\mathrm{d}F)_{ij} \in \mathcal{E}'(\mathbb{R}^n)$.*

Proof. The normal operator has the expression

$$(N_1 F)_i = \sum_{j=1}^n \frac{2x_i x_j}{|x|^{n+1}} * F_j. \tag{28}$$

Rewrite the kernel as

$$\frac{2x_i x_j}{|x|^{n+1}} = \frac{2}{n-1} \Big(\delta_{ij} |x|^{1-n} - \partial_i (x_j |x|^{1-n}) \Big) \tag{29}$$

which implies that

$$(N_1 F)_i = \frac{2}{n-1} \Big(\frac{1}{2} N_0 F_i - \sum_{j=1}^n x_j |x|^{1-n} * \partial_i F_j \Big). \tag{30}$$

Calculating the components of $\mathrm{d}(N_1F)$ we obtain

$$\partial_k(N_1F)_i - \partial_i(N_1F)_k = \frac{1}{n-1}N_0(\partial_k F_i - \partial_i F_k). \tag{31}$$

This means that $\mathrm{d}(N_1F) = (n-1)^{-1}N_0(\mathrm{d}F)$ where N_0 acts componentwise on $\mathrm{d}F$, giving the claim ☐

Now we are ready to prove our results. We start with the main theorem.

Proof of theorem 1.1. Let $f \in \mathcal{A}_V$ and $s \in (-n/4, \infty) \setminus \mathbb{Z}$. This means that $f \in H^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}$ and $P(D)f|_V = 0$ for some constant coefficient partial differential operator $P(D)$ of order $m \in \mathbb{N}$ and nonempty open set $V \subset \mathbb{R}^n$. In particular, we have $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $\hat{f} = \langle \cdot \rangle^r g$ where $g \in L^2(\mathbb{R}^n)$. Using the properties of the Fourier transform we see that $P(D)((-\Delta)^s f) = (-\Delta)^s(P(D)f)$ because $P(D)$ has constant coefficients. Since $P(D)$ is a local operator we obtain the conditions $P(D)f|_V = (-\Delta)^s(P(D)f)|_V = 0$. Now $P(D)\colon H^r(\mathbb{R}^n) \to H^{r-m}(\mathbb{R}^n)$ is continuous (see e.g. [28, Theorem 12.7]) and we have $P(D)f \in H^{r-m}(\mathbb{R}^n)$. We can use lemma 4.1 for $P(D)f$ to obtain that $P(D)f = 0$ as a tempered distribution. Taking the Fourier transform this is equivalent to that $P(\xi)\hat{f}(\xi) = P(\xi)\langle \xi \rangle^{-r} g(\xi) = 0$ almost everywhere where $P(\xi)$ is a multivariate polynomial of order $m \in \mathbb{N}$. Since $\langle \cdot \rangle^{-r} \neq 0$ everywhere and $P(\xi) \neq 0$ almost everywhere by lemma 4.3, we have that $g = 0$ almost everywhere. This implies that $\hat{f} = 0$ as a tempered distribution and hence $f = 0$.

Let then $f \in \mathscr{O}'_C(\mathbb{R}^n) \cap \mathcal{A}_V$ and $s \in (-n/2, \infty) \setminus \mathbb{Z}$. Using the same arguments as above we obtain that $P(D)f|_V = (-\Delta)^s(P(D)f)|_V = 0$ for some constant coefficient partial differential operator $P(D)$ and nonempty open set $V \subset \mathbb{R}^n$. We know that $f \in \mathscr{O}'_C(\mathbb{R}^n)$ is equivalent to that $\hat{f} \in \mathscr{O}_M(\mathbb{R}^n)$. Now $\mathcal{F}(P(D)f)(\xi) = P(\xi)\hat{f}(\xi)$ where $P(\xi)$ is a multivariate polynomial of order $m \in \mathbb{N}$. It follows from the Leibnitz product rule for multivariable functions that $\mathcal{F}(P(D)f) \in \mathscr{O}_M(\mathbb{R}^n)$ since $P(\xi)$ is polynomial and the derivatives of $\hat{f}$ are polynomially growing. This is equivalent to that $P(D)f \in \mathscr{O}'_C(\mathbb{R}^n)$ and we can use lemma 4.1 to deduce that $P(D)f = 0$ as a tempered distribution. Taking the Fourier transform this is equivalent to that $P(\xi)\hat{f}(\xi) = 0$ almost everywhere. As a polynomial $P(\xi) \neq 0$ almost everywhere and we obtain that $\hat{f} = 0$ almost everywhere. But $\hat{f}$ is continuous and hence $\hat{f} = 0$, implying $f = 0$. ☐

The rest of the results are then direct consequences of theorem 1.1.

Proof of theorem 1.3. If $f \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ or $f \in C_\infty(\mathbb{R}^n) \cap \mathcal{A}_V$, then also $f \in \mathscr{O}'_C(\mathbb{R}^n) \cap \mathcal{A}_V$. Since $N_0 = (-\Delta)^{-1/2}$ up to a constant factor and $n \geq 2$ we have that $-1/2 \in (-n/2, \infty) \setminus \mathbb{Z}$ and we can use theorem 1.1 to obtain that $f = 0$. ☐

Proof of theorem 1.4. The assumption $X_0 f = 0$ on all lines intersecting V implies that $N_0 f|_V = 0$. Since we also assume that $f \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ or $f \in C_\infty(\mathbb{R}^n) \cap \mathcal{A}_V$ we obtain $f = 0$ by theorem 1.3. ☐

Proof of theorem 1.7. The assumption $X_1F = 0$ on all lines intersecting V implies that $N_1F|_V = 0$. By lemma 4.4 we have $\mathrm{d}(N_1F) = N_0(\mathrm{d}F)$ componentwise up to a constant factor. The locality of the exterior derivative implies that $(\mathrm{d}F)_{ij} \in \mathcal{E}'(\mathbb{R}^n)$ and $N_0(\mathrm{d}F)_{ij}|_V = 0$. Since $(\mathrm{d}F)_{ij} \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{A}_V$ we can use theorem 1.3 for the components $(\mathrm{d}F)_{ij}$ to obtain that $\mathrm{d}F = 0$. Lemma 4.2 implies that $F = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$. □

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