GAUGE FREEDOMS IN THE ANISOTROPIC ELASTIC DIRICHLET-TO-NEUMANN MAP

JOONAS ILMAVIRTA

Department of Mathematics and Statistics, University of Jyväskylä

HJØRDIS SCHLÜTER

Department of Mathematics and Statistics, University of Jyväskylä

Abstract. We address the inverse problem of recovering the stiffness tensor and density of mass from the Dirichlet-to-Neumann map. We study the invariance of the Euclidean and Riemannian elastic wave equation under coordinate transformations. Furthermore, we present gauge freedoms between the parameters that leave the elastic wave equations invariant. We use these results to present gauge freedoms in the Dirichlet-to-Neumann map associated to the Riemannian elastic wave equation.

1. Introduction

We study the elastic wave equation (EWE) in the n-dimensional Euclidean space and on an n-dimensional Riemannian manifold. We address the inverse problem of recovering the anisotropic stiffness tensor and the density of mass from the Dirichlet-to-Neumann (DN) map. In this context it is essential for the reconstruction procedure whether the DN map determines the stiffness tensor and density uniquely. For that purpose we study invariance of the elastic wave equation: How the EWE behaves under coordinate transformations and what gauge freedoms there are between the parameters. Based on these results we present gauge freedoms for the DN map in the Riemannian setting and conjecture that these are the correct and full gauge groups. If this holds true then the Euclidean DN map determines the stiffness tensor and density uniquely in two (and possibly higher) dimensions. This is of particular interest as the three-dimensional Euclidean EWE is the natural setting in seismology, where the EWE models seismic waves.

Let $M \subset \mathbb{R}^n$ be the closure of a smooth domain, and let g be a Riemannian metric on M. On M we use the Euclidean coordinates x. The material parameters are the stiffness tensor c (contravariant of rank 4) and the density ρ. The stiffness tensor has the symmetry

$$c^{ijk\ell} = c^{jik\ell} = c^{k\ell ij}, \tag{1}$$

and is positive in the sense that[1]

$$c^{ijk\ell} A_{ij} A_{k\ell} \geq \delta g^{jk} g^{i\ell} A_{ij} A_{k\ell} \tag{2}$$

E-mail addresses: joonas.ilmavirta@jyu.fi , hjordis.a.schluter@jyu.fi.

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[1]Here and throughout the paper we use the Einstein summation convention, also in the Euclidean setting where all indices are kept down.

for some $\delta > 0$ and for all symmetric matrices A. See [Hoo+23] for a discussion on these structures. The density is assumed positive in the usual sense. The metric tensor is not a property of the material but of the space itself, and it is therefore forced to be Euclidean in practice — this will play an important role.

These material parameters give rise to the elastic Laplacian Δ_c^g defined by

$$(\Delta_c^g\, u(t,x))^i = \frac{1}{\sqrt{\det(g)}}\partial_{x^j}\left(\sqrt{\det(g)}c^{ijk\ell}g_{\ell m}\partial_{x^k}u^m(t,x)\right) \tag{3}$$

which maps vector fields to vector fields. The elastic wave operator is

$$P_{(\rho,c,g)} = \partial_t^2 - \rho^{-1}\Delta_c^g\,.$$

The displacement vector $u(t,x) \in \mathbb{R}^n$ satisfies the Riemannian version $P_{(\rho,c,g)}u = 0$ of the elastic wave equation.

We are therefore naturally led to the boundary value problem

$$\begin{cases} P_{(\rho,c,g)}u(t,x) = 0 & \text{in } M_T = (0,T)\times M,\\ u = f & \text{on } (0,T)\times\partial M,\\ u(0,x) = \partial_t u(0,x) = 0 & \text{in } M \end{cases} \tag{4}$$

in the spacetime $M \times (0,T)$.

For the special case $g = g_E$ the Riemannian EWE reduces to the familiar Euclidean EWE

$$\begin{cases} (P_{(\rho,c,g_E)}u(t,x)) = 0 & \text{in } M_T = (0,T)\times M\\ u = f & \text{on } (0,T)\times\partial M,\\ u(0,x) = \partial_t u(0,x) = 0 & \text{in } M. \end{cases} \tag{5}$$

The Euclidean elastic Laplacian acts as

$$(\Delta_c^{g_E}\, u(t,x))_i = \partial_{x_j}\left(c_{ijk\ell}\,\partial_{x_k}u_\ell(t,x)\right).$$

For the Euclidean elastic wave operator $P_{(\rho,c,g_E)}$ we will use the short hand notation $P_{(\rho,c,g_E)} = P_{(\rho,c)}$ in the following.

1.1. **The Finsler metric arising from elasticity.** Inverse problems related to elasticity are mainly concerned with recovering the stiffness tensor c or the reduced stiffness tensor $a = \rho^{-1}c$ everywhere from boundary data. One way to define elastic geometry is in terms of travel time distance between two points. Here one considers an elastic body that is modeled as a manifold and distance is measured by the shortest time it takes for an elastic wave to go from one point to the other. When the elastic material is elliptically anisotropic the elastic geometry is Riemannian, but this puts very stringent restrictions on the stiffness tensor.

Recent research is devoted to the fully anisotropic setting for the stiffness tensor, where only the physically necessary assumptions as in (1)–(2) are needed. The resulting elastic geometry is not Riemannian but Finslerian [Hoo+21; Hoo+19]. The Finsler metric can be derived from the principal behavior of the elastic wave operator $P_{(\rho,c,g)}$ defined above. The components of its matrix-valued principal symbol $\sigma(t,x,\omega,p)$ are

$$\sigma(P_{(\rho,c,g)})^i{}_m(t,x,\omega,\xi) = \delta^i{}_m\omega^2 - \rho^{-1}(x)c^{ijk\ell}(x)g_{\ell m}\xi_j\xi_k,$$

with $(t,x,\omega,\xi) \in T^*((0,T)\times M)$. Introducing the slowness vector $p = \omega^{-1}\xi$ and the Christoffel matrix $\Gamma(x,p)$ defined by

$$\Gamma^i{}_m(x,p) := \rho^{-1}(x)c^{ijk\ell}(x)g_{\ell m}p_jp_k,$$

one can rewrite the principal symbol as

$$\sigma(P_{(\rho,c,g)})(t,x,\omega,p) := \omega^2[I - \Gamma(x,p)].$$

By symmetry and positivity of c in (1)–(2) the Christoffel matrix Γ is symmetric and positive definite. Eigenvectors of Γ are called polarizations and the corresponding eigenvalues are related to the wave speeds. It turns out that the Finsler metric F_a associated to the fastest polarization and the density-normalized stiffness tensor field $a = \rho^{-1}c$ for the elastic geometry can be derived from the largest eigenvalue of Γ as described in [Hoo+19].

1.2. **Dirichlet-to-Neumann maps and inverse problems.** The Dirichlet-to-Neumann (DN) map is the map that sends the Dirichlet boundary condition f to its corresponding Neumann boundary condition. For the elastic wave operator $P_{(\rho,c,g)}$ defined in (4) the associated DN operator $\Lambda_{(\rho,c,g)}$ reads

$$(\Lambda_{(\rho,c,g)}f)^i = \sqrt{\det(g)}\,\nu_j\, c^{ijk\ell} g_{\ell m}\partial_{x^k}u^m\Big|_{(0,T)\times\partial M}, \quad i = 1,\dots,n, \tag{6}$$

where ν is the unit outward normal to ∂M. The inverse problem is to recover ρ, c and g from $\Lambda_{(\rho,c,g)}$ (or only ρ and c when g is assumed known).

In the Euclidean setting the DN operator $\Lambda_{(\rho,c)}$ associated to $P_{(\rho,c)}$ defined in (5) simplifies to

$$(\Lambda_{(\rho,c)}f)_i = \nu_j\, c_{ijk\ell}\,\partial_{x_k}u_\ell\Big|_{(0,T)\times\partial M}, \quad i = 1,\dots,n. \tag{7}$$

The inverse problem is to recover ρ and c from $\Lambda_{(\rho,c)}$.

This inverse problem has been addressed for the case when the stiffness tensor is isotropic [Rac00a; Rac00b; HNZ17] and for the case when the stiffness tensor is transversely isotropic or orthorhombic with additional assumptions on the symmetry axis and symmetry planes [HNZ19]. Very recent results address the reconstruction procedure for anisotropic stiffness tensors [Hoo+23].

For the DN maps corresponding to the Calderón problems for conductivities and Riemannian and Lorentzian metrics, see Appendix A.1.

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2. Results

All material parameters, metric tensors, and diffeomorphisms are assumed to be C^∞ throughout the paper. The only possibly non-smooth function is the displacement field u.

2.1. **Dirichlet-to-Neumann maps.** The point is to identify the correct gauge freedom in the inverse boundary value problems posed above. The natural gauges appear to be surprisingly different in Euclidean and Riemannian geometries, differing not only by changes of coordinates.

See section 2.2 for the definitions of the various pushforwards.

Theorem 2.1. *Let (M,g) be a Riemannian manifold of any dimension $n \geq 2$ with boundary, let c be a stiffness tensor satisfying the symmetry and positivity conditions in* (1)–(2)*, and let $\rho > 0$ be the density of mass. Let $\phi\colon M \to M$ be a diffeomorphism fixing the boundary and let $\mu \neq 0$ be a function so that $\mu|_{\partial M} = 1$. Then the DN map defined in* (6) *satisfies*

$$\Lambda_{\left(\mu^{\frac{n}{2+n}}\,\phi_*\rho,\mu\,\phi_*c,\mu^{-\frac{2}{2+n}}\,\phi_*g\right)} = \Lambda_{(\rho,c,g)}.$$

We find it likely that the gauge freedom found in the theorem is indeed the whole gauge:

Conjecture 2.1. *In the setting of Theorem 2.1 with generic stiffness tensor fields c_i, density fields ρ_i and Riemannian metrics g_i the following holds: If $\Lambda_{(\rho_1,c_1,g_1)} = \Lambda_{(\rho_2,c_2,g_2)}$, then $(\rho_1, c_1, g_1) = \left(\mu^{\frac{n}{2+n}}\,\phi_*\rho_1, \mu\,\phi_*c, \mu^{-\frac{2}{2+n}}\,\phi_*g\right)$ for some function μ and diffeomorphism ϕ.*

In the Euclidean setting we expect no gauge freedom to remain:

Conjecture 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, smooth, connected, and simply connected domain. Let $\rho_i > 0$ be density fields and c_i be stiffness tensor fields that satisfy the symmetry and positivity conditions in* (1)–(2) *for $i = 1, 2$. If the DN maps defined in* (7) *satisfy $\Lambda_{(\rho_1,c_1)} = \Lambda_{(\rho_2,c_2)}$, then $\rho_1 = \rho_2$ and $c_1 = c_2$.*

Conjecture 2.2 follows from conjecture 2.1 as follows: In the Euclidean setting $g_1 = g_2 = g_E$, so the metric component of the conclusion of conjecture 2.1 becomes $g_E = \mu^{-2/(2+n)}\phi_*g_E$. This makes ϕ a conformal map of $(\bar{\Omega}, g_E)$ fixing the boundary, which in fact forces ϕ to be the identity map[2] and $\mu = 1$. Then the rest of the components of the conclusion of conjecture 2.1 read $\rho_1 = \rho_2$ and $c_1 = c_2$ as desired.

Both conjectures are supported by the following recent observation:

Theorem 2.2 ([IN23])**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded, smooth, connected, and simply connected domain. There is an open and dense set W of density-normalized stiffness tensor fields on Ω so that if $a \in W$ and $\phi^*F_a^{qP} = F_b^{qP}$ with a diffeomorphism $\phi\colon \bar{\Omega} \to \bar{\Omega}$ fixing the boundary, then $\phi = \mathrm{id}$ and $b = a$.*

The core of the proof of theorem 2.2 is to show that ϕ has to be conformal. Then it follows from fixing the boundary that it has to be the identity.

This result suggests that the principal behaviour determines $a = \rho^{-1}c$; cf. part (3) of Proposition 2.5. Information about the density is contained in lower order terms; cf. Lemma 2.6.

This line of thought should be compared with Rachele's work [Rac00a; Rac00b; Rac03] in the isotropic[3] setting:

1. The DN map determines the travel times between boundary points for both wave speeds c_P and c_S.
2. If the manifolds $(\bar{\Omega}, c_{P/S}^{-2}g_E)$ are simple, then this geometric information determines the Riemannian metrics uniquely.
3. The metrics are known to be conformally Euclidean, which eliminates the coordinate gauge freedom intrinsic to a Riemannian manifold. Thus the DN map determines c_P and c_S.
4. The density ρ is a lower order term and can be recovered from the DN map via a ray transform.

2.2. **Various pushforwards.** We define various different kinds of pushforwards, some of which are somewhat non-standard. Let $U, V \subset \mathbb{R}^n$ be any open sets and let $\phi\colon U \to V$ be a diffeomorphism.

[2]Suppose there is a conformal map $\phi\colon \bar{\Omega} \to \bar{\Omega}$ fixing the boundary. By the Riemann mapping theorem there is a conformal map $\eta\colon D \to \Omega$ from the unit disc, and by smoothness of Ω it extends to a map $\bar{\eta}\colon \bar{D} \to \bar{\Omega}$. Thus $\bar{\eta}^{-1} \circ \phi \circ \bar{\eta}\colon \bar{D} \to \bar{D}$ is a conformal map of the disc fixing its boundary. But the conformal self-maps of the disc are the Möbius transformations, and it is easily checked that only the trivial one fixes the boundary. Thus $\bar{\eta}^{-1} \circ \phi \circ \bar{\eta} = \mathrm{id}_D$ and $\phi = \mathrm{id}_\Omega$.

[3]The stiffness tensor of an isotropic material enjoys more symmetry than we assumed, and it can be parametrized by the two Lamé parameters λ and μ as $c_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$. The wave speeds for pressure and shear waves are $c_P = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_S = \sqrt{\mu/\rho}$.

For the stiffness tensor $c^{ijk\ell}$, the metric tensor g_{ij} and the density ρ we define the pushforwards $\phi_* c$, $\phi_* g$, and $\phi_* \rho$ in the usual fashion:

$$(\phi_* c)^{\hat{i}\hat{j}\hat{k}\hat{\ell}} = \left(c^{ijk\ell}(D\phi)^{\hat{i}}{}_i (D\phi)^{\hat{j}}{}_j (D\phi)^{\hat{k}}{}_k (D\phi)^{\hat{\ell}}{}_\ell\right) \circ \phi^{-1}, \tag{8}$$

$$(\phi_* g)_{\hat{i}\hat{j}} = \left(g_{ij} (D\phi^{-1})^i{}_{\hat{i}} (D\phi^{-1})^j{}_{\hat{j}}\right) \circ \phi^{-1}, \tag{9}$$

$$\phi_* \rho = \rho \circ \phi^{-1}. \tag{10}$$

All the familiar formulas from differential geometry hold for these standard pushforwards — or, equivalently, pullbacks over ϕ^{-1}.

When the map ϕ is conformal with respect to the Euclidean metric, we define

$$(\phi_\square c)^{\hat{i}\hat{j}\hat{k}\hat{\ell}} = \left(\det(D\phi)^{-1-2/n} c^{ijk\ell}(D\phi)^{\hat{i}}{}_i (D\phi)^{\hat{j}}{}_j (D\phi)^{\hat{k}}{}_k (D\phi)^{\hat{\ell}}{}_\ell\right) \circ \phi^{-1}, \tag{11}$$

$$\phi_\square \rho = \left(\det(D\phi)^{-1} \rho\right) \circ \phi^{-1} \tag{12}$$

and

$$(\phi_\blacksquare c)^{\hat{i}\hat{j}\hat{k}\hat{\ell}} = \left(\det(D\phi)^{-2/n} c^{ijk\ell}(D\phi)^{\hat{i}}{}_i (D\phi)^{\hat{j}}{}_j (D\phi)^{\hat{k}}{}_k (D\phi)^{\hat{\ell}}{}_\ell\right) \circ \phi^{-1}. \tag{13}$$

It is important to note that the usual pushforward ϕ_* is defined on the triplet (ρ, c, g), the pushforward $\phi_\square$ on the pair (ρ, c) and pushforward $\phi_\blacksquare$ only on c.

Remark 2.3. *If we want to define a pushforward over a possibly non-conformal diffeomorphism ϕ, then the definitions of $\phi_\square$ and $\phi_\blacksquare$ need to be adjusted. The correct formulas are obtained by treating the stiffness tensor as the mixed rank object $c^{ijk}{}_\ell$ and applying the usual formulas from differential geometry. Then the EWE is indeed invariant, but the issue is that the so obtained $\phi_\square c$ no longer satisfies the symmetry conditions. For a conformal map ϕ this pushforward does preserve symmetry, and it gives rise exactly to our equation* (11)*.*

One way to see this is as follows: The usual Euclidean EWE is not coordinate invariant because it is tied to the Euclidean concept of distance. Conformal coordinate invariance only follows because Lemma 2.6 allows shifting scalar factors within the triplet (ρ, c, g).

Remark 2.4. *The elastic wave operator* (3) *on a Riemannian manifold can be written as*

$$(\Delta_c^g u(t,x))^i = \nabla_{x^j} \left(\sqrt{\det(g)} c^{ijk\ell} g_{\ell m} \nabla_{x^k} u^m(t,x)\right)$$

using covariant derivatives. These two formulas are equivalent; the only difference is in the coordinate representation of the covariant divergence.

2.3. **Invariance of the elastic wave equation.** The following results concern local behavior of the EWE where the boundary is disregarded. These results address both the Euclidean and Riemannian setting and the global results can be derived from these.

By Theorem 2.2, in most cases any diffeomorphism that preserves the Finsler metric arising from elasticity in dimension two has to be conformal. In the following proposition we write down explicitly how the Euclidean pushforward $\phi_\blacksquare c$ by a diffeomorphism $\phi\colon M \to M$ is defined so that the principal behavior of the Euclidean EWE is preserved.

Proposition 2.5. *The Euclidean and Riemannian elastic wave equations have the following invariance properties:*

(1) $P_{(\rho,c,g)} = \phi^* P_{(\phi_*\rho, \phi_* c, \phi_* g)} \phi_*$ for any diffeomorphism ϕ.
(2) $P_{(\rho,c)} = \phi^* P_{(\phi_\square \rho, \phi_\square c)} \phi_*$ for any conformal map ϕ.
(3) $\sigma(P_{(\rho,c)}) = \sigma(P_{(1,\rho^{-1}c)}) = \phi^* \sigma(P_{(1,\phi_\blacksquare(\rho^{-1}c))}) \phi_*$ for any conformal map ϕ.

The pullbacks ϕ^* and pushforwards ϕ_* surrounding the operators simply correspond to how the solution vector fields transform.

We only prove that in the Euclidean setting conformal diffeomorphisms preserve the desired structure of the stiffness tensor. For the other direction — that only conformal maps preserve such the symmetry — we refer to the theorems and conjectures of section 2.1.

Proposition 2.5 is largely based on the following lemma on scaling freedoms which may also be of independent interest:

Lemma 2.6. *The operator Δ^g_c satisfies the following relationship for the scalar functions μ and λ which satisfy $\mu \neq 0$ and $\lambda \neq 0$:*

$$(\lambda\mu\rho)^{-1}(\Delta^{\lambda g}_{\mu c})^i{}_m = \rho^{-1}(\Delta^g_c)^i{}_m + Q^i{}_m,$$

where

$$Q^i{}_m = \lambda^{-\frac{2+n}{2}}\mu^{-1}\rho^{-1}c^{ijk\ell}g_{\ell m}\partial_{x^j}\left(\mu\lambda^{\frac{2+n}{2}}\right)\partial_{x^k}.$$

Table 1 lists the consequences of lemma 2.6, where we distinguish between principal and full behavior of the PDE and between the Euclidean and the Riemannian setting. For the full behavior we list the case when $\mu\lambda^{\frac{2+n}{2}}$ is a constant so that $Qu = 0$.

	Euclidean EWE	Riemannian EWE
Principal behavior	One conformal freedom μ: $(\rho_2, c_2) \sim (\mu\rho_1, \mu c_1)$	Two conformal freedoms μ, λ: $(\rho_2, c_2, g_2) \sim (\lambda\mu\rho_1, \mu c_1, \lambda g_1)$
Full behavior	No conformal freedom	One conformal freedom μ: $(\rho_2, c_2, g_2) \sim (\mu^{\frac{n}{2+n}}\rho_1, \mu c_1, \mu^{-\frac{2}{2+n}}g_1)$

TABLE 1. Consequences of lemma 2.6 for inverse problems. The relation $\sim$ means that the two parameters give rise to the same boundary data on $\partial\Omega$ or ∂M. These are scaling freedoms, not diffeomorphism freedoms; the freedom to choose coordinates in the Euclidean setting is eliminated by the demand that boundary be fixed by the diffeomorphism.

3. Proofs

3.1. Proofs of auxiliary results.

Proof of Lemma 2.6. A calculation gives

$$\begin{aligned}
&(\lambda\mu\rho)^{-1}\left(\Delta^{\lambda g}_{\mu c}u\right)^i \\
&\quad= \lambda^{-1}\mu^{-1}\rho^{-1}\frac{1}{\sqrt{\det(\lambda g)}}\partial_{x^j}\left(\sqrt{\det(\lambda g)}\,\mu\, c^{ijk\ell}\,\lambda\, g_{\ell m}\,\partial_{x^k}u^m\right) \\
&\quad= \underbrace{\rho^{-1}\frac{1}{\sqrt{\det(g)}}\sum_{j=1}^n \partial_{x^j}\left(\sqrt{\det(g)}\,c^{ijk\ell}\,g_{\ell m}\,\partial_{x^k}u^m\right)}_{=\rho^{-1}(\Delta^g_c u)^i} \\
&\qquad+ \underbrace{\lambda^{-\frac{2+n}{2}}\mu^{-1}\rho^{-1}\sum_{j=1}^n c^{ijk\ell}\,g_{\ell m}\partial_{x^j}\left(\mu\lambda^{\frac{2+n}{2}}\right)\partial_{x^k}u^m}_{=(Qu)^i}.
\end{aligned}$$

as claimed. □

Proof of Proposition 2.5. **Part (1):** All structures used in this claim are tensor fields and covariant derivatives in the sense of Riemannian geometry, and therefore they are invariant under a change of coordinates by design. Verification using equations (8), (9) and (10) by hand is also straightforward.

Part (2): A conformal map ϕ with respect to the Euclidean geometry satisfies

$$\phi_* g_E = \det(D\phi)^{-2/n} g_E.$$

Combining this with part (1) gives

$$\phi_* \left(\rho^{-1}\Delta_c^{g_E}\right) u = (\phi_*\rho)^{-1}\Delta_{\phi_* c}^{\det(D\phi)^{-2/n} g_E}\, \phi_* u$$

for any u.

We then apply Lemma 2.6 with $\lambda = \det(D\phi)^{-2/n}$ and $\mu = \det(D\phi)^{1+2/n}$ (so that $\mu\lambda^{n/2+1} = 1$) to get

$$(\phi_*\rho)^{-1}\Delta_{\phi_* c}^{\det(D\phi)^{-2/n} g_E} = \det(D\phi)(\phi_*\rho)^{-1}\Delta^{g_E}_{\det(D\phi)^{-1-2/n}\phi_* c}.$$

The definitions in (11) and (12) were set up so that $\phi_\square c = \det(D\phi)^{-1-2/n}\phi_* c$ and $\phi_\square \rho = \det(D\phi)^{-1}\phi_*\rho$. Therefore

$$\rho^{-1}\Delta_c^{g_E}\, u = \phi^* \left((\phi_\square\rho)^{-1}\Delta^{g_E}_{\phi_\square c}\right)\phi_* u$$

from which the desired claim follows.

Part (3): This is similar to part (2) above, but we no longer need to satisfy $\mu\lambda^{n/2+1} = 1$ as the subprincipal term of Lemma 2.6 is irrelevant. This corresponds to the principal symbol only depending on the density normalized stiffness tensor $\rho^{-1}c$, which is evident. Our definition in (13) satisfies

$$\phi_\blacksquare c = (\phi_\square 1)^{-1}\phi_\square c$$

with the constant density 1, so the claim follows from essentially the same calculation as above. □

3.2. **Proof of the main result.**

Proof of Theorem 2.1. As emphasized in part (1) of Proposition 2.5, the pushforwards of c, g and ρ defined in (8)–(10) and the pushforward $\phi_* u$ of the solution leave the Riemannian EWE invariant with respect to any diffeomorphism ϕ. Therefore for any diffeomorphism ϕ that fixes the boundary the triplet $(\phi_*\rho, \phi_* c, \phi_* g)$ gives rise to the same DN map as the triplet (ρ, c, g):

$$\Lambda_{(\phi_*\rho,\phi_* c,\phi_* g)} = \Lambda_{(\rho,c,g)}. \tag{14}$$

Additionally it was shown as a consequence of Lemma 2.6 that there is the following conformal freedom for the elastic wave operator $P_{(\rho,c,g)}$ for any non-vanishing function μ:

$$P_{\left(\mu^{\frac{n}{2+n}}\rho,\mu c,\mu^{-\frac{2}{2+n}} g\right)} = P_{(\rho,c,g)}.$$

This implies for any function $\mu \neq 0$ that satisfies $\mu|_{\partial M} = 1$:

$$\Lambda_{\left(\mu^{\frac{n}{2+n}}\rho,\mu c,\mu^{-\frac{2}{2+n}} g\right)} = \Lambda_{(\rho,c,g)}. \tag{15}$$

Combining (14) and (15) yields

$$\Lambda_{\left(\mu^{\frac{n}{2+n}}\phi_*\rho,\mu\,\phi_* c,\mu^{-\frac{2}{2+n}}\phi_* g\right)} = \Lambda_{(\rho,c,g)}$$

as claimed. □

Appendix A. Relation to Euclidean, Riemannian, and Lorentzian Calderón problems

Coordinate gauge freedom and conformal freedoms occur in some other inverse boundary value problems, and we discuss some examples to provide a point of comparison to our results.

The citations given in this appendix are only examples of the vast literature on these problems. For more details and background, we refer the reader to [Uhl13; Bel07; Las18; KKL01] and references therein.

A.1. DN maps for Euclidean, Riemannian, and Lorentzian Calderón problems.

The inverse problems we introduced in section 1.2 are closely related to the Calderón problem concerned with the conductivity equation describing electrostatics:

$$\begin{cases} L_\gamma u(x) := \nabla \cdot (\gamma(x)\nabla u(x)) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where γ denotes an isotropic or anisotropic conductivity. The DN map corresponding to L_γ is defined by

$$\Lambda_\gamma f = \nu_i\, \gamma_{ij}\, \partial_{x_j} u \Big|_{\partial\Omega}. \tag{16}$$

The Calderón problem is then concerned with recovering γ from Λ_γ. For an anisotropic conductivity γ it turns out that this inverse problem is closely related to a corresponding geometric inverse problem. Consider the Dirichlet problem associated to the Laplace–Beltrami operator

$$\begin{cases} \Delta_g u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where (cf. Remark 2.4 for an alternative formula)

$$\Delta_g u = \frac{1}{\sqrt{\det(g)}} \partial_{x^i} \left(\sqrt{\det(g)} g^{ij} \partial_{x^j} u \right)$$

and with corresponding DN map defined by

$$\Lambda_g f = \nu_i\, g^{ij}\, \partial_{x^j} u \sqrt{\det g} \Big|_{\partial\Omega}.$$

Then the geometric Calderón problem asks to recover g from Λ_g. One can also consider the following geometric version of the conductivity equation on a Riemannian manifold (M, g):

$$\begin{cases} \operatorname{div}_g(\beta \nabla_g u) = 0 & \text{in } M \\ u = f & \text{on } \partial M, \end{cases}$$

where β is an isotropic conductivity. In this setting the DN map is defined as

$$\Lambda_{(\beta,g)} f = \nu_i\, \beta\, g^{ij}\, \partial_{x^j} u \sqrt{\det g} \Big|_{\partial\Omega}. \tag{17}$$

This gives rise to the Calderón problem on a Riemannian manifold that asks to recover β and g from $\Lambda_{(\beta,g)}$.

There is also a similarity to the Lorentzian Calderón problem in that there is a conformal gauge freedom in all dimensions. An example is the non-linear wave equation

$$\begin{cases} \Box_g u(x) + a(x)u(x)^4 = 0, & \text{on } M, \\ u(x) = f(x), & \text{on } \partial M, \\ u(t,x') = 0, & t < 0, \end{cases}$$

where $\Box_g$ is the d'Alembert operator of g (the Laplace–Beltrami operator on a Lorentzian manifold) and $M = \mathbb{R} \times N$. The corresponding DN map is defined as

$$\Lambda_{(g,a)} f = \nu^j \partial_{x^j} u|_{(0,T)\times\partial N} \tag{18}$$

and the Lorentzian Calderón problem asks to recover the metric g and the function a from $\Lambda_{(g,a)}$.

A.2. **Gauge freedom in the Euclidean and Riemannian Calderón problems.** It was observed by L. Tartar that the map Λ_γ defined in (16) does not determine γ uniquely (see [KV84] for an account). This is due to the fact that any C^∞ diffeomorphism $\phi\colon \overline{\Omega} \to \overline{\Omega}$ with $\phi|_{\partial\Omega} = \text{id}$ and with the Euclidean pullback of the conductivity

$$\phi^\Box \gamma = \frac{1}{\det(D\phi)} D\phi\, \gamma\, D\phi^t, \tag{19}$$

produces the same DN map as γ:

$$\Lambda_{\phi^\Box\gamma} = \Lambda_\gamma.$$

Notice that the pullback defined in (19) and the pushforward defined in (11) are not the usual pullbacks and pushforwards of tensor fields in differential geometry.

Similarly to the Euclidean case Λ_g as defined in (A.1) does not determine g uniquely due to the coordinate gauge freedom

$$\Lambda_{\phi_* g} = \Lambda_g,$$

where ϕ_* denotes the classical pushforward of g by ϕ. It was shown in [LU01] that this is the only gauge freedom in dimension $n \geq 3$.

In dimension $n = 2$ the Laplace–Beltrami operator is conformally invariant which gives rise to an additional gauge for the inverse problem:

$$\Lambda_{\alpha(\phi_* g)} = \Lambda_g.$$

This is proven by [LU01] to be the only obstruction for unique identifiability of g.

The inverse problems for the DN maps of (16) and (A.1) are equivalent when $n \geq 3$. Namely, if we set

$$g = (\det\gamma)^{\frac{1}{n-2}}\gamma^{-1} \quad \text{or equivalently}$$
$$\gamma = (\det\, g)^{\frac{1}{2}} g^{-1},$$

then (see [LU89])

$$\Lambda_g = \Lambda_\gamma.$$

A.3. **Gauge freedom in the Calderón problem on manifolds.** It was shown in [SU03] that the conformal freedom of the Laplace–Beltrami operator in two dimensions yields the following conformal freedom for the DN operator defined in (17) in two dimensions for $\phi\colon M \to M$ with $\phi|_{\partial M} = \text{id}$:

$$\Lambda_{(\alpha(\phi_* g), \phi_*\beta)} = \Lambda_{(g,\beta)},$$

for any positive scalar function α.

A.4. **Gauge freedom in the Lorentzian Calderón problem.** Similarly to the previous Calderón problems there is a gauge freedom by a diffeomorphism for the DN map defined in (18). It was shown in [HUZ22] that for $\phi\colon M \to M$ with $\phi|_{\partial M} = \mathrm{id}$ we have

$$\Lambda_{(\alpha(\phi^* g), \phi^* a)} = \Lambda_{(g,a)}$$

and that there is the conformal freedom

$$\Lambda_{(e^{-2\beta} g, e^{-\beta} a)} = \Lambda_{(g,a)}$$

for a smooth function β on M such that

$$\beta|_{\partial M} = 0, \quad \partial_\nu \beta|_{\partial M} = 0, \quad \Box_g e^{-\beta} = 0.$$

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