

BROKEN RAY TOMOGRAPHY IN THE DISK

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ABSTRACT. Given a bounded C^1 domain $\Omega \subset \mathbb{R}^n$ and a nonempty subset E of its boundary (set of tomography), we consider broken rays which start and end at points of E . We ask: If the integrals of a function over all such broken rays are known, can the function be reconstructed? We give positive answers when Ω is a ball and the unknown function is required to be uniformly quasianalytic in the angular variable and the set of tomography is open. We also analyze the situation when the set of tomography is a singleton.

1. INTRODUCTION

Consider a ray of light travelling to the right on the real axis. If the intensity of the light at a point x is $I(x)$ and the material on the real axis has a non-constant attenuation coefficient $f \geq 0$, the differential equation

$$(1) \quad I'(x) = -f(x)I(x)$$

is satisfied. Given the initial intensity $I(0)$ and the intensity at a distance x , we may easily find (under suitable regularity assumptions) that

$$(2) \quad \log \left(\frac{I(0)}{I(x)} \right) = \int_0^x f(y) dy.$$

That is, by measuring the ratio of initial and final intensity of a beam of light we in effect measure the integral of the attenuation coefficient over its trajectory. This is true even when the trajectory is more complicated than a segment of the real line.

The fundamental problem of (scalar) ray tomography is to deduce knowledge of the attenuation coefficient f from its integrals over a suitable set of trajectories. Different materials (and different wavelengths) have different attenuation coefficients, so information about f can be translated into understanding of the structure of the object being imaged.

Consider a bounded C^1 domain Ω in a Euclidean space and a continuous attenuation coefficient $f : \bar{\Omega} \rightarrow \mathbb{R}$. A piecewise linear path in $\bar{\Omega}$ with reflections at the boundary as dictated by geometric optics (the angle of incidence equals the angle of reflection) is called a *broken ray*. We have a device E , which is a nonempty subset of the boundary $\partial\Omega$, from which we may emit light and measure the intensity of incoming light. The rest of the boundary acts as a reflector and we consider those broken rays which start and end at E , which we call the *set of tomography*. The corresponding set of integrals of f over such broken rays is called the *broken ray transform of f* with the set of tomography E . We will implicitly assume all broken rays to start and end at the set of tomography.

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We ask the following question: If the integral of f is known over all broken rays with both endpoints in E , can the function f be fully reconstructed? How does the result depend on the domain Ω , the set of tomography E , and regularity assumptions on the unknown function f ?

In the case $E = \partial\Omega$ broken rays are simply intersections of lines with the domain Ω . By extending the unknown function f by zero to $\mathbb{R}^n \setminus \bar{\Omega}$, the problem reduces to the classical X-ray transform. This transform is well understood: The answer to the uniqueness question is affirmative, and also regularity and feasible reconstruction algorithms are known. For these results we refer to the textbooks by Helgason [9] and Natterer [14] and the original works of Radon [15] and Cormack [6].

Reflections from a boundary of a domain naturally arise in connection with billiards. For details we refer to the textbooks by Tabachnikov [16] and Chernov and Markarian [5]. Reflections in connection to the X-ray transform as considered here, however, appear to have been studied less.

Florescu, Markel, and Schotland [8] have studied the broken ray Radon transform in connection with the problem of recovering absorption and scattering coefficients in single scattering tomography. Eskin [7] has studied the reconstruction of electromagnetic potential in the presence of convex reflecting obstacles. Lozev [12] has also studied the numerical aspect of scalar tomography with similar obstacles. The question asked above has also turned out to be related to the Calderón problem with partial data [11].

It should be noted that in optical tomography the signal is very weak after multiple reflections, and reconstruction schemes should be based on broken rays with only few reflections. One such method is imaging by Compton cameras [3, 4, 1]. The broken ray transform with one reflection is also known as the V-line Radon transform, which has recently been studied and applied to imaging problems [13, 17, 2].

The main results presented here require knowledge of the broken ray transform with arbitrarily many reflections, and can therefore not be expected to be practical in optical tomography. Our main motivation is the Calderón problem with partial data; on suitable manifolds, reconstructibility of a conductivity from partial Dirichlet to Neumann data can be reduced to the injectivity of the broken ray transform [11]. The broken ray transform arises similarly in partial data problems for the electromagnetic Schrödinger equation [7].

In this article we consider the special case where Ω is the unit disk $D \subset \mathbb{R}^2$, and give the following answers:

Theorem 1. *If $f : \bar{D} \rightarrow \mathbb{R}$ is continuous and the set of tomography is a singleton, then integrals over broken rays uniquely determine the integral of f over any circle centered at the origin.*

Corollary 2. *Let B^n denote the unit ball in $\mathbb{R}^n$. If $n \geq 2$, $f : \bar{B}^n \rightarrow \mathbb{R}$ is continuous, and the set of tomography is a singleton, then integrals over broken rays uniquely determine the integral of f over any circle centered at the origin with the singleton in the circle's plane.*

Full reconstruction is guaranteed for functions that are uniformly quasianalytic in the angular variable in the following theorem and its corollary. Uniformly quasianalytic functions are continuous in the radial variable and quasianalytic¹ in the

¹For example, real-analytic functions are quasianalytic.

angular variable with a uniformity condition given in definition 14. There is, in particular, great freedom in the radial dependence of the unknown function. Assuming quasianalyticity in both variables makes the problem much easier, as demonstrated in corollary 6.

Theorem 3. *If $f : \bar{D} \rightarrow \mathbb{R}$ is uniformly quasianalytic in the angular variable in the sense of definition 14 and its angular derivatives of all orders satisfy the Dini-Lipschitz condition and the set of tomography is open, then integrals over broken rays uniquely determine f everywhere in the disk.*

Corollary 4. *Theorem 3 also holds when D is replaced by the unit ball in any Euclidean space of dimension two or higher.*

We need not assume any regularity of f at the origin to guarantee reconstruction in the punctured disk $\bar{D} \setminus \{0\}$ in theorems 1 and 3. If we assume continuity at the origin, theorem 1 implies that $f(0)$ can be reconstructed.

One class of functions satisfying the regularity assumptions of theorem 3 are functions of the form (in polar coordinates)

$$(3) \quad f(r, \vartheta) = a_0(r) + \sum_{k=1}^K (a_k(r) \cos(k\vartheta) + b_k(r) \sin(k\vartheta)),$$

where $K \in \mathbb{N}$ and the functions a_k, b_k are Hölder continuous.

We also show in lemma 17 that if theorem 3 would hold for all functions which are C^∞ in the angular variable, then it would also hold for all continuous functions. Whether theorem 3 indeed holds for all continuous function remains an interesting open question.

We begin by making some general remarks about the nature of the problem and presenting the formalism for the broken ray transform in the disk (section 2). In section 2.3 we prove theorem 1 and corollary 2; singleton tomography is discussed in more detail in section 5. After developing the necessary tools in sections 3 and 4, we present the proof of theorem 3 and corollary 4 in section 6.

2. PRELIMINARIES

2.1. Some initial remarks. Let us first consider the general problem with a bounded C^1 Euclidean domain Ω . The problem is linear in the function f , so we may reformulate the uniqueness question of the introduction as follows: If the integral of f over all broken rays vanishes, is f identically zero?

Linearity also causes the problem for a complex-valued f to reduce to the real-valued case: the problems of finding the real and imaginary parts of f decouple.

Decreasing the size of E makes the problem more difficult by decreasing the number of broken rays available for reconstruction. If it is convenient to consider some trajectories that have reflections also on E , we may do so. Summing the integrals over each part of the trajectory starting and ending at E , we may construct the integral over a trajectory with reflections on E . Thus the only requirement for the broken rays of interest is that their endpoints lie in the set of tomography E .

The assumption of C^1 regularity of $\partial\Omega$ is necessary to make sense of the reflections everywhere. A natural assumption on the unknown function $f : \Omega \rightarrow \mathbb{R}$ is continuity. Corollary 6 demonstrates that too strong regularity assumptions make the problem very easy.

Convexity of Ω is not required in the following proposition, but we keep the assumption to make the proof simpler.

Proposition 5. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded convex domain. If $f : \bar{\Omega} \rightarrow \mathbb{R}$ is continuous and the integral of f over all broken rays vanishes, then f vanishes in the exterior of the convex hull of the complement of the union of the convex hulls of the components of the set of tomography E . This set is an intersection of Ω with a neighborhood of E in $\bar{\Omega}$ if Ω is strictly convex and $E \subset \partial\Omega$ is open.*

Let E_i , $i = 1, \dots, m$, be the connected components of E . The first claim is that f vanishes in the exterior of $U = \text{co}(\Omega \setminus \bigcup_{i=1}^m \text{co } E_i)$, where $\text{co } A$ is the convex hull of a set A . In dimension two $\Omega \setminus \bigcup_{i=1}^m \text{co } E_i$ is convex, so f vanishes in $\text{int } \bigcup_{i=1}^m \text{co } E_i$.

Proof of proposition 5. Let $\varepsilon > 0$ and denote by f_ε the function f convolved with the standard mollifier with support in $B(0, \varepsilon)$. The integral of f vanishes over any line that does not meet U , whence the integral of f_ε vanishes over any line that does not meet the convex set $B(U, \varepsilon) = \{x \in \mathbb{R}^n : d(x, U) < \varepsilon\}$. Since $f_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, the function f_ε vanishes outside $B(U, \varepsilon)$ by Helgason's support theorem [9, Theorem 2.6].

Take any point $y \in \text{int}(\Omega \setminus U)$. For all $\varepsilon < d(y, U)$ we know that $f_\varepsilon(y) = 0$. Since f is continuous at y , $f_\varepsilon(y)$ converges to $f(y)$ as $\varepsilon \rightarrow 0$. Thus f vanishes in the exterior of U as claimed.

For the second claim, fix any point $a \in E$. To prove the claim, we find a neighborhood V of a in the topology of $\bar{\Omega}$ such that $V \cap \Omega \subset \text{int}(\Omega \setminus U)$.

Let E_i be the component of E containing a . By convexity there is a non-degenerate linear form $L : \mathbb{R}^n \rightarrow \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that $\Omega \subset \{x \in \mathbb{R}^n : L(x) < c\}$ and $L(a) = c$. By strict convexity of Ω and openness of E there is $\varepsilon > 0$ such that for $V = \{x \in \Omega : L(x) > c - \varepsilon\}$ we have $V \cap \partial\Omega \subset E$. Since $\Omega \setminus V$ is convex and $V \subset \text{co } E_i$, we have $U \cap V = \emptyset$. But V is also open, so $V \cap \Omega \subset \text{int}(\Omega \setminus U)$. $\square$

Corollary 6. *Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded strictly convex domain and the set of tomography $E \subset \partial\Omega$ is open. If f is quasianalytic in Ω and continuous in $\bar{\Omega}$ and the integral of f over all broken rays vanishes, then f is identically zero.*

Proof. By proposition 5 f vanishes in a nonempty open subset of Ω , from which the claim follows by quasianalyticity of f . $\square$

Note that in the preceding proposition and corollary only the non-reflecting trajectories were used, although their union is not even dense in Ω .

2.2. Formalism for the broken ray transform in the disk. We use polar coordinates (r, ϑ) in the unit disk $D \subset \mathbb{R}^2$ and identify points on the boundary $S_1 = \partial D$ with the corresponding angle. The notation $E = \{0\}$ means that E is one point at angle zero.

A broken ray γ is characterized by the number of line segments n_γ (there are $n_\gamma - 1$ reflections), the initial point $\iota_\gamma \in E$, the final point $\kappa_\gamma \in E$, the angle α_γ at which a line segment appears when seen from the origin, and an integer winding number m_γ such that the trajectory condition

$$(4) \quad n_\gamma \alpha_\gamma = \kappa_\gamma - \iota_\gamma + 2\pi m_\gamma$$

is satisfied. These five parameters $(n_\gamma, m_\gamma, \alpha_\gamma, \iota_\gamma, \kappa_\gamma)$ uniquely (and redundantly) define the broken ray γ . We require that $n_\gamma > 0$ but the other parameters may

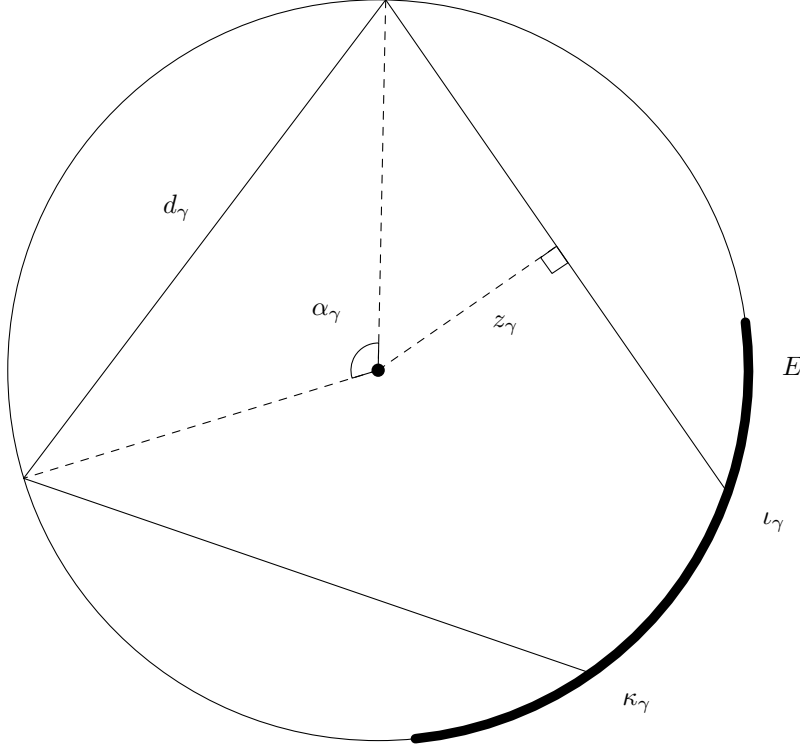


FIGURE 1. An example of a broken ray γ in the disk. Here $m_\gamma = 1$ and $n_\gamma = 3$. The parameters α_γ , z_γ , d_γ , ι_γ , and κ_γ as well as the set of tomography E are illustrated in the figure. The angles ι_γ and κ_γ are identified with the corresponding points on the boundary.

be negative. We denote by Γ_E the set of all broken rays from E to E , allowing intermediate reflections on E . We denote by $z_\gamma = \cos(\alpha_\gamma/2)$ and $d_\gamma = 2|\sin(\alpha_\gamma/2)|$ the distance from the origin to the trajectory and the length of each individual line segment. The parameters describing the broken ray are illustrated in figure 1.

We also write

$$(5) \quad \Gamma_E^N = \{\gamma \in \Gamma_E : n_\gamma \geq N\}$$

to denote the set of rays which have sufficiently many reflections.

A trajectory γ is a (piecewise linear) curve, and we denote its trace by $\text{tr}(\gamma)$.

Let $C = C(\bar{D}; \mathbb{R})$ be the space of continuous real-valued mappings from the closed unit disk, $C^{\text{DL}}(\bar{D}) \subset C$ the subspace of functions which satisfy the Dini-Lipschitz condition, and $B(\Gamma_E, \mathbb{R})$ the space of bounded real-valued mappings from Γ_E . The mappings in $B(\Gamma_E, \mathbb{R})$ need not be linear, continuous, or measurable (the set Γ_E is not equipped with any such structure), but merely bounded. We equip all of these spaces with the supremum norm. The Dini-Lipschitz condition states that the modulus of continuity ω_f of a function f satisfy $\omega_f(\delta) = o(\log^{-1} \delta^{-1})$ for small δ . Hölder continuous functions, for example, satisfy the Dini-Lipschitz condition.

Definition 7. For $f \in C$, we define the *broken ray transform of f* as the function $\mathcal{G}f : \Gamma_E \rightarrow \mathbb{R}$ such that

$$(6) \quad \mathcal{G}f(\gamma) = \oint_{\gamma} f d\mathcal{H}^1 = \frac{1}{l} \int_0^l f(\gamma(t)) dt,$$

when $\gamma : [0, l] \rightarrow \bar{D}$ is a broken ray with unit speed. The map $\mathcal{G} : C \rightarrow B(\Gamma_E, \mathbb{R})$ is the *broken ray transform*. We define the unnormalized broken ray transform $\tilde{\mathcal{G}}$ by

$$(7) \quad \tilde{\mathcal{G}}f(\gamma) = \int_{\gamma} f d\mathcal{H}^1.$$

Elementary calculations show that $\mathcal{G}$ is linear and continuous. Using a constant function as an example, one may show that moreover $\|\mathcal{G}\| = 1$. The unnormalized version, however, is not continuous for generic E : the unnormalized broken ray transform of a constant function is not bounded, as can be seen by increasing the length of the trajectory indefinitely.

2.3. Proof of theorem 1. We are now ready to prove theorem 1. A Fourier analytic proof for the special case when $f \in C^{\text{DL}}(\bar{D})$ is given in section 5, where the result is also discussed in more detail.

Proof of theorem 1. We denote circles centered at the origin by $S_r = S(0, r)$. We denote the average of f over such circles by $a_0(r) = \oint_{S_r} f d\mathcal{H}^1$. It suffices to consider those γ that have $\gcd(n_{\gamma}, m_{\gamma}) = 1$. We may assume that $\iota_{\gamma} = \kappa_{\gamma} = 0$.

Denoting $\varphi_l^{\pm}(r) = (l - \frac{1}{2})\alpha_{\gamma} \pm \arccos(z_{\gamma}/r)$, we have

$$(8) \quad \begin{aligned} \tilde{\mathcal{G}}f(\gamma) &= \sum_{l=1}^{n_{\gamma}} \int_{z_{\gamma}}^1 (f(r, \varphi_l^+(r)) + f(r, \varphi_l^-(r))) \frac{r}{\sqrt{r^2 - z_{\gamma}^2}} dr \\ &= \frac{d_{\gamma}}{2} \int_{z_{\gamma}}^1 p_{z_{\gamma}}(r) \sum_{y \in S_r \cap \text{tr}(\gamma)} f(y) dr, \end{aligned}$$

where we have defined the function $p_z : (z, 1] \rightarrow \mathbb{R}$ for any $z \in (0, 1)$ such that

$$(9) \quad p_z(r) = ((1 - z^2)(1 - z^2/r^2))^{-1/2}.$$

Thus

$$(10) \quad \mathcal{G}f(\gamma) = \frac{1}{n_{\gamma} d_{\gamma}} \tilde{\mathcal{G}}f(\gamma) = \int_{z_{\gamma}}^1 p_{z_{\gamma}}(r) \frac{1}{2n_{\gamma}} \sum_{y \in S_r \cap \text{tr}(\gamma)} f(y) dr.$$

The function $p_{z_{\gamma}}$ has a clear geometric interpretation: it is the probability distribution for the distance from the origin on the trajectory γ . Therefore we expect that $\mathcal{G}f(\gamma)$ approaches $g(z_{\gamma})$,

$$(11) \quad g(z) = \int_z^1 p_z(r) a_0(r) dr,$$

when $n_{\gamma} \rightarrow \infty$ in a suitable sense.

Let us fix some $\gamma \in \Gamma_E^3$. The set $S_r \cap \text{tr}(\gamma)$ has $2n_{\gamma}$ elements for almost every $r \in (z_{\gamma}, 1)$. For any such r we may write

$$(12) \quad S_r \cap \text{tr}(\gamma) = \{x_1^{\gamma, r}, \dots, x_{2n_{\gamma}}^{\gamma, r}\}$$

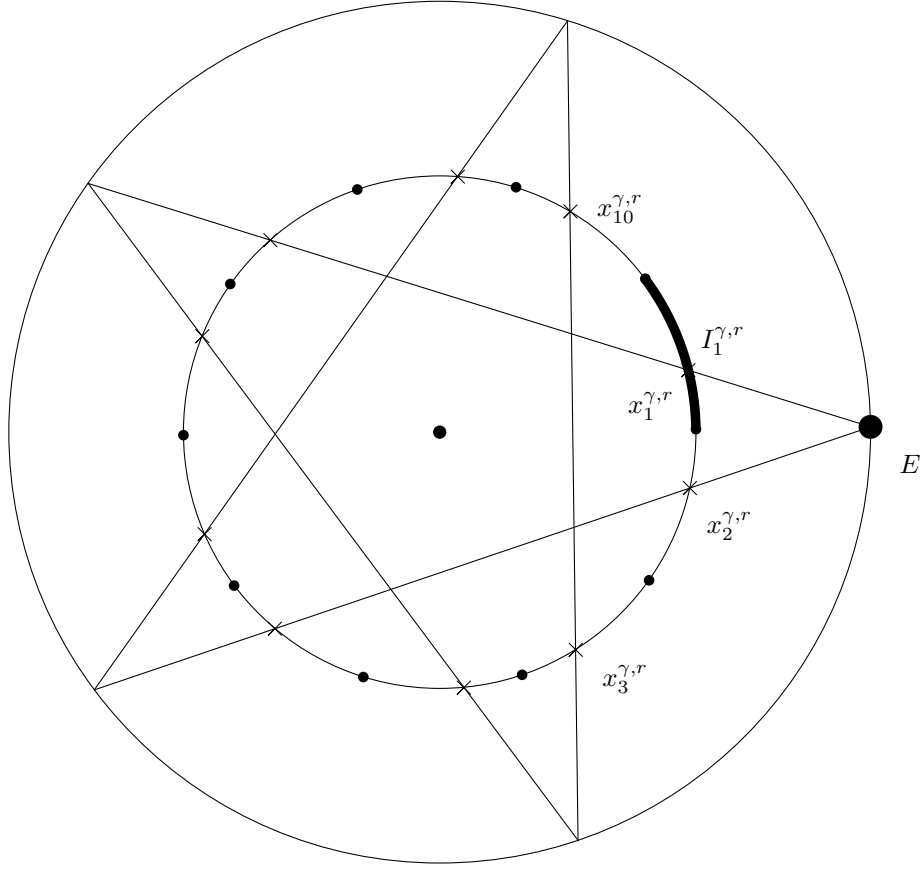


FIGURE 2. Illustration of the points $x_i^{\gamma,r}$ and arcs $I_i^{\gamma,r}$ in the proof of theorem 1. In this example $m_\gamma = 2$, $n_\gamma = 5$, and $r \approx 0.6$. The points $x_i, \dots, x_{10}$ on S_r are denoted by crosses and the endpoints of the arcs $I_i^{\gamma,r}$ by dots. For clarity, only one arc and few points are labeled. The singleton E is highlighted at the boundary ∂D . The choice of the arcs $I_i^{\gamma,r}$ presented here is only one of many possibilities.

and assume that the points $x_i^{\gamma,r}$ are numbered clockwise on S_r . We split S_r to $2n_\gamma$ pairwise disjoint arcs $I_i^{\gamma,r}$ such that each arc corresponds to the angle π/n_γ and $x_i^{\gamma,r} \in I_i^{\gamma,r}$. This can be done since the points $x_i^{\gamma,r}$ are pairwise uniformly distributed on S_r in the sense that a relabeling $x_i^{\gamma,r} \mapsto x_{i+2}^{\gamma,r}$ corresponds to a rotation by an angle $2\pi/n_\gamma$. The length of each arc $I_i^{\gamma,r}$ is $\mathcal{H}^1(I_i^{\gamma,r}) = \pi r/n_\gamma$, so the diameter is $\text{diam}(I_i^{\gamma,r}) < \pi r/n_\gamma$. The points $x_i^{\gamma,r}$ and arcs $I_i^{\gamma,r}$ are illustrated in figure 2.

Let $\varepsilon > 0$. Since f is continuous in the compact set $\bar{D}$, it is uniformly continuous and there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. If we choose

γ so that $n_\gamma > \pi/\delta$, then $\pi r/n_\gamma < \delta$ for all $r \in (z_\gamma, 1)$, whence for almost every $r \in (z_\gamma, 1)$ we find

$$(13) \quad \left| \int_{I_i^{\gamma,r}} f d\mathcal{H}^1 - f(x_i^{\gamma,r}) \right| < \varepsilon,$$

which yields

$$(14) \quad \begin{aligned} \left| a_0(r) - \frac{1}{2n_\gamma} \sum_{y \in S_r \cap \text{tr}(\gamma)} f(y) \right| &= \left| \frac{1}{2\pi r} \sum_{i=1}^{2n_\gamma} \int_{I_i^{\gamma,r}} f d\mathcal{H}^1 - \frac{1}{2n_\gamma} \sum_{i=1}^{2n_\gamma} f(x_i^{\gamma,r}) \right| \\ &= \left| \frac{1}{2n_\gamma} \sum_{i=1}^{2n_\gamma} \left(\int_{I_i^{\gamma,r}} f d\mathcal{H}^1 - f(x_i^{\gamma,r}) \right) \right| \\ &\leq \frac{1}{2n_\gamma} \sum_{i=1}^{2n_\gamma} \varepsilon = \varepsilon \end{aligned}$$

and

$$(15) \quad \begin{aligned} |\mathcal{G}f(\gamma) - g(z_\gamma)| &= \left| \int_{z_\gamma}^1 p_{z_\gamma}(r) \frac{1}{2n_\gamma} \sum_{y \in S_r \cap \text{tr}(\gamma)} f(y) dr - \int_{z_\gamma}^1 p_z(r) a_0(r) dr \right| \\ &\leq \int_{z_\gamma}^1 p_{z_\gamma}(r) \left| \frac{1}{2n_\gamma} \sum_{y \in S_r \cap \text{tr}(\gamma)} f(y) - a_0(r) \right| dr \\ &\leq \int_{z_\gamma}^1 p_{z_\gamma}(r) \varepsilon dr = \varepsilon, \end{aligned}$$

since $\int_{z_\gamma}^1 p_z(r) dr = 1$.

Comparing (11) with the zeroth Abel transform defined in (23), we have

$$(16) \quad g(z) = \frac{\mathcal{A}_0 a_0(z)}{2\sqrt{1-z^2}}.$$

Since f is continuous by assumption, also a_0 is continuous. The Abel transform preserves continuity (lemma 10), so g is continuous on $[0, 1]$. For any $r \in [0, 1]$ we can find a sequence $(\gamma_i)_{i=1}^\infty$ such that $\gamma_i \in \Gamma_E^i$ and $z_{\gamma_i} \rightarrow r$ as $i \rightarrow \infty$.

Let $\varepsilon > 0$. By the estimate (15) we can find $N = N_\varepsilon \in \mathbb{N}$ such that $|\mathcal{G}f(\gamma_i) - g(z_{\gamma_i})| \leq \frac{1}{2}\varepsilon$ whenever $i \geq N$. Since $z_{\gamma_i} \rightarrow r$ and g is continuous, we can choose N so that also $|g(z_{\gamma_i}) - g(r)| \leq \frac{1}{2}\varepsilon$ whenever $i \geq N$. By these estimates we have $|\mathcal{G}f(\gamma_i) - g(r)| \leq \varepsilon$ for $i \geq N$, whence

$$(17) \quad \lim_{i \rightarrow \infty} \mathcal{G}f(\gamma_i) = g(r)$$

for any $r \in [0, 1]$ independently of the choice of the sequence (γ_i) . Since each $\mathcal{G}f(\gamma_i)$ is known ($\mathcal{G}f$ is known), we may recover the function g from the data.

By (16) this implies that we can recover $\mathcal{A}_0 a_0$ on $[0, 1]$. Since $\mathcal{A}_0 a_0$ is continuous, we can recover it on the whole interval $[0, 1]$. The Abel transform $\mathcal{A}_0$ is injective on $C([0, 1])$ (to which space a_0 belongs by assumption) and has an explicit inversion formula (lemma 10 below), allowing us to finally reconstruct the function a_0 . $\square$

Proof of corollary 2. The geometry of the ball $\bar{B}^n$ is such that any broken ray γ lays in the intersection of a two dimensional subspace of $\mathbb{R}^n$ and the ball. One may pick any plane containing the singleton and the origin and reduce the problem to singleton tomography in $\bar{B}^2$. The result then follows trivially from theorem 1. $\square$

3. INTEGRAL TRANSFORMS

We write our function $f \in C^{\text{DL}}(\bar{D})$ as a Fourier series in the angular variable. The following lemma describes the sense in which the series represents the function. The lemma is true also for the complex Fourier series.

Lemma 8. *The Fourier series of any function $f \in C^{\text{DL}}(\bar{D})$ converges to f uniformly in the disk: There are continuous functions $a_k : [0, 1] \rightarrow \mathbb{R}$, $k \geq 0$, and $b_k : [0, 1] \rightarrow \mathbb{R}$, $k \geq 1$, such that the sequence of functions $(f_K)_K$,*

$$(18) \quad f_K(r, \vartheta) = a_0(r) + \sum_{k=1}^K (a_k(r) \cos(k\vartheta) + b_k(r) \sin(k\vartheta)),$$

converges uniformly to f in the disk as $K \rightarrow \infty$. These functions a_k and b_k admit a common (Dini-Lipschitz) modulus of continuity and satisfy $a_k(0) = b_k(0) = 0$ when $k \geq 1$.

Proof. We set for every $r \in [0, 1]$

$$(19) \quad a_0(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \vartheta) d\vartheta$$

and for $k \geq 1$

$$(20) \quad a_k(r) = \frac{1}{\pi} \int_0^{2\pi} f(r, \vartheta) \cos(k\vartheta) d\vartheta$$

and

$$(21) \quad b_k(r) = \frac{1}{\pi} \int_0^{2\pi} f(r, \vartheta) \sin(k\vartheta) d\vartheta.$$

These functions manifestly have the property that $a_k(0) = b_k(0) = 0$ when $k \geq 1$.

The functions $f(r, \cdot)$, $r \in (0, 1]$, admit a common modulus of continuity which satisfies the Dini-Lipschitz condition. (Clearly the functions a_k and b_k admit essentially the same modulus of continuity.) The Fourier series of a uniformly continuous function satisfying the Dini-Lipschitz condition converges uniformly and the rate of convergence depends only on the modulus of continuity [18, Theorem 10.3 of Chapter 2], whence each $f_K \rightarrow f$ uniformly. $\square$

Let $f \in C^{\text{DL}}(\bar{D})$. Using the notations of the above proof, $f_K \rightarrow f$ uniformly and $\mathcal{G}$ is continuous in the corresponding topology, whence $\mathcal{G}f_K \rightarrow \mathcal{G}f$ uniformly. This allows us to study the Fourier components a_k and b_k of f one by one; this will be done in lemma 12 and its corollary.

What we will be able to construct from the broken ray transform $\mathcal{G}f$ is some integral transforms of its Fourier components a_k and b_k as defined in lemma 8. Since the case $k = 0$ reduces to the classical Abel transform, we call these transforms generalized Abel transforms. These transforms appear implicitly in Cormack's approach [6] to the Radon transform in the plane.

Definition 9. For $k = 1, 2, \dots$ we define the k th generalized Abel transform as a map $\mathcal{A}_k : C_0([0, 1]) \rightarrow C([0, 1])$ such that

$$(22) \quad \mathcal{A}_k f(z) = 2 \int_z^1 T_k(z/y) \frac{f(y)}{\sqrt{1 - (z/y)^2}} dy,$$

where T_k is the k th Chebyshev polynomial and $C_0([0, 1]) = \{f \in C([0, 1]) : f(0) = 0\}$.

For $k = 0$ we define the *zeroth generalized Abel transform* (the Abel transform up to a change of variable) as a map $\mathcal{A}_0 : C([0, 1]) \rightarrow C([0, 1])$ such that

$$(23) \quad \mathcal{A}_0 f(z) = 2 \int_z^1 \frac{f(y)}{\sqrt{1 - (z/y)^2}} dy.$$

Since $T_0 \equiv 1$, (22) is correct also in the case $k = 0$.

We summarise the main properties of these transforms. Some of the properties are based on those of the Radon transform in the plane, which are covered e.g. in [9].

Lemma 10. *The generalized Abel transforms defined in definition 9 have the following properties with any $k \geq 0$:*

- (1) $\mathcal{A}_k$ indeed maps $C_0([0, 1])$ ($C([0, 1])$ for $k = 0$) into $C([0, 1])$ as defined.
- (2) $\mathcal{A}_k$ is linear.
- (3) The transform is continuous and $\|\mathcal{A}_k\| \leq 2$ when $C([0, 1])$ is equipped with the supremum norm.²
- (4) $\mathcal{A}_k$ is injective.
- (5) We have the explicit inversion formula

$$(24) \quad f(y) = -\frac{1}{\pi} \frac{d}{dy} \int_y^1 T_k(z/y) \frac{\mathcal{A}_k f(z)}{z \sqrt{(z/y)^2 - 1}} dz.$$

Proof. (1) For $f \in C([0, 1])$ let $g_k(r, \vartheta) = f(r) \cos(k\vartheta)$. Also, let $\mathcal{R}$ be the Radon transform parametrized so that

$$(25) \quad \mathcal{R}g_k(\rho, \varphi) = \int_{L_{\rho, \varphi}} g_k d\mathcal{H}^1,$$

where $L_{\rho, \varphi} = \{x \in \mathbb{R}^2 : x_1 \cos \varphi + x_2 \sin \varphi = \rho\}$.

A simple calculation yields

$$(26) \quad \mathcal{R}g_k(\rho, \varphi) = \mathcal{A}_k f(\rho) \cos(k\varphi).$$

Since $f(0) = 0$ if $k > 0$, g_k is continuous, and so is $\mathcal{R}g_k$. Therefore the above equation shows that $\mathcal{A}_k f$ is continuous.

(2) Trivial.

(3) The result follows from the estimate $|T_k(x)| \leq 1$ and the observation

$$(27) \quad \int_z^1 \frac{1}{\sqrt{1 - (z/y)^2}} dy = \sqrt{1 - z^2}.$$

²Using Hölder's inequality it is not difficult to show that $\mathcal{A}_k : L^p([0, 1]) \rightarrow L^q([0, 1])$ is continuous for any $p \in (2, \infty]$ and $q \in [1, \infty]$.

(4) The Radon transform is injective on continuous functions, so the observation made in part (1) of the proof shows that $\mathcal{A}_k$ is injective. We also have the inversion formula

$$(28) \quad f(r) = \mathcal{R}^{-1}[(\rho, \varphi) \mapsto \cos(k\varphi)\mathcal{A}_k f(\rho)](r, 0)$$

using the inverse Radon transform.

(5) See Eqs. (10) and (18) in [6]. The treatment therein only requires that f is continuous. $\square$

4. TOOLS FOR OPEN SET TOMOGRAPHY

4.1. Explicit form for the broken ray transform. Using the Fourier series representation given in lemma 8 and the generalized Abel transforms we can explicitly calculate the broken ray transform in terms of the Fourier components. Lemma 12 gives the basic result for each Fourier component, and its corollary lists the relevant special cases we will need.

To simplify further notation, we define the coefficient

$$(29) \quad S_k(\gamma) = \begin{cases} \frac{\sin(k(\kappa_\gamma - \iota_\gamma)/2)}{\sin(k\alpha_\gamma/2)} & \text{when } k\alpha_\gamma \notin 2\pi\mathbb{Z} \\ n_\gamma(-1)^{(n_\gamma+1)\frac{k\alpha_\gamma}{2\pi} + km_\gamma} & \text{when } k\alpha_\gamma \in 2\pi\mathbb{Z} \end{cases}$$

for all $k = 0, 1, \dots$ and $\gamma \in \Gamma_E$. This coefficient has the following properties.

Lemma 11. *For any k and γ we have:*

- (1) $|S_k(\gamma)| \leq n_\gamma$.
- (2) *The sum identities*

$$(30) \quad \begin{aligned} \sum_{l=1}^{n_\gamma} \cos(k(l-1/2)\alpha_\gamma) &= S_k(\gamma) \cos(k(\kappa_\gamma - \iota_\gamma)/2) \quad \text{and} \\ \sum_{l=1}^{n_\gamma} \sin(k(l-1/2)\alpha_\gamma) &= S_k(\gamma) \sin(k(\kappa_\gamma - \iota_\gamma)/2) \end{aligned}$$

hold.

- (3) *If $(\kappa_\gamma - \iota_\gamma)/\pi$ or α_γ/π is irrational, $S_k(\gamma) \neq 0$.*

Proof. (1) By (4)

$$(31) \quad \sin(k(\kappa_\gamma - \iota_\gamma)/2) = \sin(kn_\gamma\alpha_\gamma/2 - km_\gamma\pi) = (-1)^{km_\gamma} \sin(kn_\gamma\alpha_\gamma/2),$$

so it suffices to show that the function

$$(32) \quad f(x) = \frac{\sin(nx)}{\sin(x)}$$

satisfies $|f(x)| \leq n$ for all $x \in \mathbb{R} \setminus \pi\mathbb{Z}$ when $n \in \mathbb{N}$ and $n \geq 2$.

If we define $f(m\pi) = n(-1)^{(n-1)m}$ for all $m \in \mathbb{Z}$, the function f becomes real analytic on the real line. In particular, we have $f(0) = n$.

By symmetry, it is enough to show that $|f(x)| \leq n$ for all $x \in [0, \pi/2]$. To this end we wish to show that $f(0) = n$, $f' \neq 0$ on $(0, \pi/2n)$, and $|f(x)| \leq n$ when $x \in [\pi/2n, \pi/2]$.

Indeed, if $f'(x) = 0$, then $n \tan(x) = \tan(nx)$, but this equation has no solutions on $(0, \pi/2n)$. And since the sine function is concave on $[0, \pi/2]$, we have $\sin(\pi/2n) \geq 1/n$, whence $1/\sin(x) \leq n$ for all $x \in [\pi/2n, \pi/2]$. This yields the desired estimate.

(2) Taking real and imaginary parts of the geometric sum

$$(33) \quad \sum_{l=1}^n e^{ik(l-1/2)\alpha} = \begin{cases} \frac{\frac{i}{2}(1 - e^{ik\beta})}{\sin(k\alpha/2)} & \text{when } e^{ik\alpha} \neq 1 \\ ne^{ik\alpha/2} & \text{when } e^{ik\alpha} = 1, \end{cases}$$

when α and β satisfy $n\alpha = \beta + 2\pi m$ for an integer m , immediately gives the result.

(3) By (4) $(\kappa_\gamma - \iota_\gamma)/\pi$ is irrational if and only if α_γ/π is. The conclusion follows thus immediately from the definition of $S_k(\gamma)$. $\square$

Lemma 12. *Suppose $\gamma \in \Gamma_E$ is such that $\iota_\gamma = 0$. If $f(r, \vartheta) = a_k(r) \cos(k\vartheta) + b_k(r) \sin(k\vartheta)$, where $k = 0, 1, \dots$ and the functions a_k and b_k are continuous, then*

$$(34) \quad \tilde{\mathcal{G}}f(\gamma) = S_k(\gamma)(\cos(k\kappa_\gamma/2)\mathcal{A}_k a_k(z_\gamma) + \sin(k\kappa_\gamma/2)\mathcal{A}_k b_k(z_\gamma)).$$

Proof. Let us only consider the case $b_k = 0$; the calculations for the b_k term are essentially the same.

Fix some $\gamma \in \Gamma_E$ with $\iota_\gamma = 0$. For any radius $r \in [0, 1]$ and direction $\varphi \in \partial D$ we define the corresponding secant $H_{r, \varphi} = \{y \in D : \varphi \cdot y = r\}$. The trajectory γ is composed of n_γ such secants. Each of these secants corresponds to the radius $r = z_\gamma$. We identify the directions φ with angles and label them by $\varphi_l = (l-1/2)\alpha_\gamma$, where $l = 1, 2, \dots, n_\gamma$.

After a change of the variable, we find

$$(35) \quad \int_{H_{z_\gamma, \varphi_l}} f d\mathcal{H}^1 = \int_{-\alpha_\gamma/2}^{\alpha_\gamma/2} a_k \left(\frac{\cos(\alpha_\gamma/2)}{\cos(\varphi)} \right) \cos(k(\varphi_l + \varphi)) \frac{\cos(\alpha_\gamma/2)}{\cos^2(\varphi)} d\varphi.$$

Since

$$(36) \quad \cos(k(\varphi_l + \varphi)) = \cos(k\varphi_l) \cos(k\varphi) - \sin(k\varphi_l) \sin(k\varphi)$$

and the second term vanishes in the integral, another change of variable $r = z_\gamma / \cos(\varphi)$ gives

$$(37) \quad \begin{aligned} \int_{H_{z_\gamma, \varphi_l}} f d\mathcal{H}^1 &= \cos(k\varphi_l) \int_{-\alpha_\gamma/2}^{\alpha_\gamma/2} a_k \left(\frac{\cos(\alpha_\gamma/2)}{\cos(\varphi)} \right) \cos(k\varphi) \frac{\cos(\alpha_\gamma/2)}{\cos^2(\varphi)} d\varphi \\ &= 2 \cos(k\varphi_l) \int_0^{\alpha_\gamma/2} a_k \left(\frac{z_\gamma}{\cos(\varphi)} \right) T_k(\cos(\varphi)) \frac{1}{\sin(\varphi)} \frac{z_\gamma \sin(\varphi)}{\cos^2(\varphi)} d\varphi \\ &= 2 \cos(k\varphi_l) \int_{z_\gamma}^1 a_k(r) T_k(z_\gamma/r) \frac{1}{\sqrt{1 - (z_\gamma/r)^2}} dr \\ &= \cos(k\varphi_l) \mathcal{A}_k a_k(z_\gamma). \end{aligned}$$

The secants H_{z_γ, φ_l} , $l = 1, \dots, n_\gamma$, intersect only at a finite number of points and $\text{tr}(\gamma) = \bigcup_{l=1}^{n_\gamma} H_{z_\gamma, \varphi_l}$, so

$$(38) \quad \tilde{\mathcal{G}}f(\gamma) = \sum_{l=1}^{n_\gamma} \cos(k\varphi_l) \mathcal{A}_k a_k(z_\gamma).$$

Using lemma 11 to evaluate the sum, we arrive at the claim. $\square$

Corollary 13. *Let f be as in lemma 12. If $0 \in E$ and $\gamma \in \Gamma_E$ is such that $\iota_\gamma + \kappa_\gamma = 0$, then*

$$(39) \quad \tilde{\mathcal{G}}f(\gamma) = S_k(\gamma) \mathcal{A}_k a_k(z_\gamma).$$

In particular, if $E = \{0\}$ so that $\iota_\gamma = \kappa_\gamma = 0$, for injectivity it is sufficient to consider only those γ for which $\gcd(n_\gamma, m_\gamma) = 1$. In this case

$$(40) \quad \tilde{\mathcal{G}}f(\gamma) = n_\gamma(-1)^{km_\gamma/n_\gamma} \mathcal{A}_k a_k(z_\gamma)$$

if k/n_γ is an integer and $\tilde{\mathcal{G}}f(\gamma) = 0$ otherwise.

Proof. The function f may be written as a component of a Fourier series in the variable $\tilde{\vartheta} = \vartheta - \iota_\gamma$:

$$(41) \quad f(r, \tilde{\vartheta}) = \tilde{a}_k(r) \cos(k\tilde{\vartheta}) + \tilde{b}_k(r) \sin(k\tilde{\vartheta}).$$

This corresponds to choosing coordinates so that $\iota_\gamma = 0$, so that the problem reduces to the situation in lemma 12. Due to elementary trigonometric identities the coefficients of this Fourier series relate to the ones in lemma 8 so that

$$(42) \quad \begin{aligned} a_k(r) &= \tilde{a}_k(r) \cos(k\iota_\gamma) - \tilde{b}_k(r) \sin(k\iota_\gamma) \quad \text{and} \\ b_k(r) &= \tilde{a}_k(r) \sin(k\iota_\gamma) + \tilde{b}_k(r) \cos(k\iota_\gamma). \end{aligned}$$

Lemma 12 indicates now that

$$(43) \quad \begin{aligned} \tilde{\mathcal{G}}f(\gamma) &= S_k(\gamma)(\cos(k\kappa_\gamma)\mathcal{A}_k \tilde{a}_k(z_\gamma) + \sin(k\kappa_\gamma)\mathcal{A}_k \tilde{b}_k(z_\gamma)) \\ &= S_k(\gamma)(\cos(k\iota_\gamma)\mathcal{A}_k \tilde{a}_k(z_\gamma) - \sin(k\iota_\gamma)\mathcal{A}_k \tilde{b}_k(z_\gamma)) \\ &= S_k(\gamma)\mathcal{A}_k a_k(z_\gamma), \end{aligned}$$

which is the first part of the claim.

Let us now turn to the case $E = \{0\}$. Given γ , let $\tilde{\gamma}$ be such that $n_{\tilde{\gamma}} = n_\gamma / \gcd(n_\gamma, m_\gamma)$, $m_{\tilde{\gamma}} = m_\gamma / \gcd(n_\gamma, m_\gamma)$, $\alpha_{\tilde{\gamma}} = \alpha_\gamma$, and $\text{tr}(\tilde{\gamma}) = \text{tr}(\gamma)$. It is geometrically obvious that γ contains $\gcd(n_\gamma, m_\gamma)$ copies of $\tilde{\gamma}$ and so $\mathcal{G}f(\gamma) = \mathcal{G}f(\tilde{\gamma})$. Thus it suffices to consider those $\gamma \in \Gamma_{\{0\}}$ with coprime n_γ and m_γ .

Now that $n_\gamma \alpha_\gamma = 2\pi m_\gamma$ and $\gcd(n_\gamma, m_\gamma) = 1$, $k\alpha_\gamma \in 2\pi\mathbb{Z}$ if and only if k/n_γ is an integer. Thus the definition of $S_k(\gamma)$ together with the result $\tilde{\mathcal{G}}f(\gamma) = S_k(\gamma)\mathcal{A}_k a_k(z_\gamma)$ gives the second part of the claim. $\square$

Note that the functions b_k do not appear in the above formulas for $\mathcal{G}f(\gamma)$. This is due to the assumption that the broken ray is symmetric with respect to the normal to ∂D at angle 0, which causes functions antisymmetric with respect to this line to integrate to zero over the trajectory.

4.2. Quasianalytic functions. The class of functions for which our argument shows the uniqueness result of theorem 3 consists of functions uniformly quasianalytic in the angular variable as defined below. There is, however, no implication that this is the optimal class in which the theorem should hold.

Definition 14. Let $M = (M_n)_{n=0}^\infty$ be an increasing logarithmically convex sequence³ of strictly positive real numbers. A sequence $(a_k)_{k=1}^\infty$ of real numbers is defined to be in the class $S^\#(M)$ if there is a constant $R > 0$ such that for each n it satisfies

$$(44) \quad \sum_{k=0}^{\infty} k^{2n} |a_k|^2 \leq M_n R^n.$$

³The sequence $(M_n)_{n=0}^\infty$ is logarithmically convex if the sequence $(\log M_n)_{n=0}^\infty$ is convex.

The class $S^\#(M)$ is called a *quasianalytic class of sequences* if the defining sequence M satisfies

$$(45) \quad \sum_{n=0}^{\infty} \sqrt{\frac{M_n}{M_{n+1}}} = \infty.$$

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ with a period 2π belongs to the class $C^\#(M)$ if it can be written as a uniformly convergent Fourier series as

$$(46) \quad f(x) = \sum_{k=0}^{\infty} (a_k \cos(kx) + b_k \sin(kx))$$

such that the sequences $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=1}^{\infty}$ are in the class $S^\#(M)$. If the class $S^\#(M)$ is quasianalytic, then $C^\#(M)$ is a *quasianalytic class of functions*. Functions in such a class are *quasianalytic functions*.

If a continuous function $f : \bar{D} \rightarrow \mathbb{R}$ is written as a Fourier series as in lemma 8, it is *uniformly quasianalytic in the angular variable* if for each $R \in (0, 1]$ the collection of functions $\{f(r, \cdot) : r \geq R\}$ belong to the same quasianalytic class.

In dimensions higher than 2, we define a function $f : \bar{B}^n \rightarrow \mathbb{R}$, $n \geq 3$, to be uniformly quasianalytic in the angular variables if its restriction to any two dimensional plane intersecting the origin is uniformly quasianalytic in the sense defined in two dimensions.

The class $C^\#(M)$ of 2π -periodic smooth functions is a quasianalytic class of functions in the sense defined above if and only if it is so in the classical sense: if f is in a quasianalytic class and the function and all of its derivatives vanish at any point, then the function is identically zero. For this result and more discussion on periodic quasianalytic functions we refer to [10, Section V.2]. We only remark here that quasianalytic functions are smooth and all derivatives of the Fourier series (which can be taken term by term) converge uniformly to the corresponding derivative of the function.

Angular differentiation of a function uniformly quasianalytic in the angular variable may also be carried out term by term and the series converges uniformly, provided that the angular derivatives of all orders satisfy the Dini-Lipschitz condition. This is why we make the assumption in theorem 3.

The important property of quasianalytic functions that motivates the use of this class is the following lemma.

Lemma 15. *Let $(x_k)_{k=0}^{\infty}$ be a bounded sequence and $z \in (0, 1)$. If a uniformly quasianalytic function f is written as a Fourier series as in lemma 8 and*

$$(47) \quad \sum_{k=0}^{\infty} (-k^2)^n x_k \mathcal{A}_k a_k(z) = 0$$

for all $n = 0, 1, \dots$, then $x_k \mathcal{A}_k a_k(z) = 0$ for each k .

Proof. The functions $a_k(r, \cdot)$, $r \geq z$, belong to the same quasianalytic class $C^\#(M)$. By lemma 10(3) we have $(\mathcal{A}_k a_k(z))_{k=0}^{\infty} \in S^\#(M)$.

Define a function $g : \mathbb{R} \rightarrow \mathbb{R}$ so that

$$(48) \quad g(x) = \sum_{k=0}^{\infty} x_k \mathcal{A}_k a_k(z) \cos(kx).$$

Now g is quasianalytic and obviously its odd derivatives vanish at zero. By assumption also the function itself and its even derivatives vanish at zero, so g must be identically zero. This proves the lemma. $\square$

5. TOMOGRAPHY FROM A SINGLETON

We prove theorem 1 in the special case when the unknown function f satisfies the Dini-Lipschitz condition. This result is not needed for the proof of the general case (which was presented in section 2.3), but we present it since it is simple and uses the tools developed for an open set of tomography. Both proofs are constructive and the reconstructions are based on the ‘long trajectory limit’, where n_γ tends to infinity.

Proposition 16. *For $f \in C^{\text{DL}}(\bar{D})$, $\mathcal{G}f : \Gamma_E \rightarrow \mathbb{R}$ determines the zeroth Fourier component $a_0(r) = \int_{S(0,r)} f d\mathcal{H}^1$ when E is a singleton.*

Proof. As noted in corollary 13, it is sufficient to consider those $\gamma \in \Gamma_E$ for which $\gcd(n_\gamma, m_\gamma) = 1$. By continuity of f it is enough to reconstruct it outside the boundary ∂D .

Let a_k and b_k be as in lemma 8. Let $g(r, \vartheta) = a_0(r)$ and

$$(49) \quad f_K(r, \vartheta) = \sum_{k=1}^K (a_k(r) \cos(k\vartheta) + b_k(r) \sin(k\vartheta)).$$

By corollary 13 $\mathcal{G}g(\gamma) = d_\gamma^{-1} \mathcal{A}_0 a_0(z_\gamma)$, and using this we show that $\mathcal{G}f(\gamma) - \mathcal{G}g(\gamma) \rightarrow 0$ as $n_\gamma \rightarrow \infty$.

Let $\varepsilon > 0$. By lemma 8 there is K_ε such that $\|f - g - f_{K_\varepsilon}\| < \varepsilon$. By corollary 13 $\mathcal{G}f_{K_\varepsilon}(\gamma) = 0$ whenever $\gamma \in \Gamma_E^{K_\varepsilon+1}$, whence

$$(50) \quad |\mathcal{G}f(\gamma) - \mathcal{G}g(\gamma)| = |\mathcal{G}(g + f_{K_\varepsilon} - f)(\gamma) - \mathcal{G}f_{K_\varepsilon}(\gamma)| \leq \varepsilon + 0.$$

This estimate holds for all $\gamma \in \Gamma_E^{K_\varepsilon+1}$.

Fix any $w \in [0, 1)$. There is a sequence $(\gamma_i)_{i \in \mathbb{N}}$ such that $n_{\gamma_i} \geq i$, and $z_{\gamma_i} \rightarrow w$ (and thus $d_{\gamma_i} \rightarrow 2\sqrt{1-w^2}$) as $i \rightarrow \infty$. The estimate obtained above gives thus $\lim_{i \rightarrow \infty} \mathcal{G}g(\gamma_i) = \frac{1}{2}(1-w^2)^{-1/2} \mathcal{A}_0 a_0(w)$, allowing one to reconstruct the function $\mathcal{A}_0 a_0$ on $[0, 1)$. Injectivity of the Abel transform (lemma 10) proves the claim. $\square$

By corollary 13 and theorem 1 we have the following for $f \in C^{\text{DL}}(\bar{D})$ and a singleton E (chosen to be at angle zero): $\mathcal{G}f|_{\Gamma_E} = 0$ if and only if $a_0 = 0$ and

$$(51) \quad \sum_{n_\gamma | k} (-1)^{km_\gamma/n_\gamma} \mathcal{A}_k a_k(z_\gamma) = 0$$

for all $\gamma \in \Gamma_E$ with $\gcd(n_\gamma, m_\gamma) = 1$. Here the sum is over all k with $n_\gamma | k$.

First, nothing can be said about the functions b_k . Furthermore, there are a number of functions a_k such that $\mathcal{A}_k a_k(z_\gamma) = 0$ whenever $n_\gamma | k$; by injectivity $\mathcal{A}_k(C_0([0, 1]))$ is an infinite dimensional subspace of $C([0, 1])$, and the space $\{f \in \mathcal{A}_k(C_0([0, 1])) : f(z_\gamma) = 0 \text{ when } n_\gamma | k\}$ has finite codimension in $\mathcal{A}_k(C_0([0, 1]))$ (for each k , there are only finitely many γ with $n_\gamma | k$) and therefore infinite dimension. Hence the condition $\mathcal{G}f|_{\Gamma_E} = 0$ leaves great freedom to the functions a_k , $k \geq 1$, and we can efficiently recover only a_0 from $\mathcal{G}f|_{\Gamma_E}$.

One can only possibly image the unknown function f in the set $\bigcup_{\gamma \in \Gamma_E} \text{tr}(\gamma)$. When E is countable, this set is dense in the disk but has Hausdorff dimension one, so without continuity assumptions no recovery results can be expected.

6. TOMOGRAPHY FROM AN OPEN SET

We denote by $C^{n,m} = C_{\text{rad}}^n C_{\text{ang}}^m$ the space of real valued functions on the punctured disk $\bar{D}^* = \bar{D} \setminus \{0\}$ which have n continuous partial derivatives with respect to the radial and m to the angular variable. The angular derivative is denoted by ∂_{ang} . We require no regularity at the origin and correspondingly restrict $\mathcal{G}f$ to $\Gamma_E^* = \{\gamma \in \Gamma_E : 0 \notin \text{tr}(\gamma)\}$ for $f \in C^{n,m}$. Including or excluding the origin in the domain of the unknown function does not alter the results of theorems 1 and 3 besides reconstructibility at the origin.

Let a rotation operator $\mathfrak{R}_\varphi$, $\varphi \in \mathbb{R}$, act on functions so that $\mathfrak{R}_\varphi f(r, \vartheta) = f(r, \vartheta + \varphi)$. Let another rotation operator $\mathfrak{R}_\varphi$ act on trajectories instead of functions so that $\iota_{\mathfrak{R}_\varphi \gamma} = \iota_\gamma + \varphi$, $\kappa_{\mathfrak{R}_\varphi \gamma} = \kappa_\gamma + \varphi$ and other parameters describing γ are unaltered. Geometrically $\mathfrak{R}_\varphi$ simply rotates the trajectory by an angle φ .

We can only show the uniqueness result of theorem 3 for functions uniformly quasianalytic in the angular variable, but if it would hold for functions in $C^{0,\infty}$, the following lemma implies that it would also hold for the class $C^{0,0}$.

Lemma 17. *For $E \subset \partial D$ open and any n , the following hold true:*

- (1) *If the uniqueness result of theorem 3 would hold for the class $C^{n,\infty}$, then it would also hold for $C^{n,0}$.*
- (2) *If $f \in C^{n,m}$ and $\mathcal{G}f(\gamma) = 0$ for all $\gamma \in \Gamma_E^*$, then also $\mathcal{G}\partial_{\text{ang}}^i f(\gamma) = 0$ for all $\gamma \in \Gamma_E^*$ and $i \in \mathbb{N}$, $i \leq m$.*
- (3) *Suppose $f \in C^{0,0}$ and $\mathcal{G}f(\gamma) = 0$ for all $\gamma \in \Gamma_E^*$. Fix $\gamma \in \Gamma_E^*$ and $\delta > 0$ such that $\mathfrak{R}_\varphi \gamma \in \Gamma_E^*$ whenever $|\varphi| < \delta$. Then $\mathcal{G}\mathfrak{R}_\varphi f(\gamma) = 0$ for all $|\varphi| < \delta$.*

Proof. Each part is based on the following observation. The change of the integral of f over a trajectory is the same if the function is rotated clockwise as if the trajectory is rotated counterclockwise by the same angle φ :

$$(52) \quad \mathcal{G}f(\mathfrak{R}_{-\varphi} \gamma) = \mathcal{G}\mathfrak{R}_\varphi f(\gamma).$$

(1) Let us denote a shrunk set of tomography by

$$(53) \quad E^\delta = \{a \in E : d(x, \partial D \setminus E) > \delta\}.$$

For a sufficiently small $\delta > 0$ the set E^δ is nonempty and open.

Let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ be the standard mollifier and let $\eta_\varepsilon(x) = \varepsilon^{-1} \eta(x/\varepsilon)$. Let $\varepsilon < \delta$.

If $f \in C^{n,0}$ is such that $\mathcal{G}f|_{\Gamma_E^*} = 0$, then $\mathcal{G}f(\mathfrak{R}_\varphi \gamma) = 0$ for all $\gamma \in \Gamma_{E^\delta}^*$ and $|\varphi| < \delta$. Thus for any $\gamma \in \Gamma_{E^\delta}^*$ we have

$$(54) \quad \begin{aligned} 0 &= \int_{-\infty}^{\infty} \eta_\varepsilon(\varphi) \mathcal{G}f(\mathfrak{R}_\varphi \gamma) d\varphi \\ &= \int_{-\infty}^{\infty} \eta_\varepsilon(\varphi) \mathcal{G}\mathfrak{R}_{-\varphi} f(\gamma) d\varphi \\ &= \mathcal{G}(\eta_\varepsilon * f)(\gamma), \end{aligned}$$

where the convolution is with respect to the angular variable of f . Therefore $\mathcal{G}f|_{\Gamma_E^*} = 0$ implies that $\mathcal{G}(\eta_\varepsilon * f)|_{\Gamma_{E^\delta}^*} = 0$. If $f \in C^{n,0}$, then $\eta_\varepsilon * f \in C^{n,\infty}$.

If theorem 3 holds in the class $C^{n,\infty}$, then $\mathcal{G}(\eta_\varepsilon * f)|_{\Gamma_{E^\delta}^*} = 0$ implies that $\eta_\varepsilon * f = 0$. Since this holds for all $\varepsilon \in (0, \delta)$ and $\eta_\varepsilon * f \rightarrow f$ uniformly as $\varepsilon \rightarrow 0$, we find that $f = 0$.

(2) Suppose $m \geq 1$ and $f \in C^{n,m}$. It is enough to show the result for the first derivative, from which the argument can be iterated.

Dividing (52) by φ and passing to the limit $\varphi \rightarrow 0$ we obtain

$$(55) \quad \partial_\varphi \mathcal{G}f(\Re_\varphi \gamma)|_{\varphi=0} = -\mathcal{G}\partial_{\text{ang}}f(\gamma)$$

for all $\gamma \in \Gamma_E^*$. Since $\mathcal{G}f$ vanishes on Γ_E^* , this implies that $\mathcal{G}\partial_{\text{ang}}f$ does, too.

(3) Trivial. $\square$

Proof of theorem 3. We show that $\mathcal{G}f(\gamma) = 0$ for all $\gamma \in \Gamma_E$ implies $f = 0$ if f is uniformly quasianalytic in the angular variable and satisfies the Dini-Lipschitz condition.

Let f be written as a Fourier series as in lemma 8. By theorem 1 we already know that $a_0 = 0$. Differentiating term by term, we have

$$(56) \quad \partial_{\text{ang}}^{2n}f(r, \vartheta) = \sum_{k=0}^{\infty} (-k^2)^n (a_k(r) \cos(k\vartheta) + b_k(r) \sin(k\vartheta))$$

for any $n \in \mathbb{N}$. We may choose coordinates so that the angle zero lies in E . Fix any $\gamma \in \Gamma_E$ such that $\iota_\gamma + \kappa_\gamma = 0$; such a trajectory can be found since E is open.

By corollary 13, lemma 17 and the assumption of vanishing broken ray transform we have

$$(57) \quad \sum_{k=0}^{\infty} (-k^2)^n S_k(\gamma) \mathcal{A}_k a_k(z_\gamma) = 0$$

for $n = 0, 1, \dots$. Hence by lemmas 11 and 15 we have that $S_k(\gamma) \mathcal{A}_k a_k(z_\gamma) = 0$.⁴

If $\iota_\gamma = -\kappa_\gamma \notin \pi\mathbb{Q}$, we have $S_k(\gamma) \neq 0$ for all k by lemma 11 and so $\mathcal{A}_k a_k(z_\gamma) = 0$ for all k . The set of such z_γ that $\iota_\gamma = -\kappa_\gamma \notin \pi\mathbb{Q}$ and $\gamma \in \Gamma_E$ is dense on $[0, 1]$, so continuity of $\mathcal{A}_k a_k$ (lemma 10) implies that $\mathcal{A}_k a_k = 0$. This, by the injectivity of $\mathcal{A}_k$ (lemma 10), implies that each a_k is identically zero.

We have thus demonstrated that the function f is antisymmetric with respect to the normal to ∂D at angle zero. The same conclusion may be drawn with any line intersecting E and the origin. If f is antisymmetric with respect to two different lines going through the origin, f vanishes everywhere. $\square$

Notice that by proposition 5 (in dimension two) $f \in C$ with $\mathcal{G}f = 0$ vanishes in $\bar{D} \setminus \text{co}(\partial D \setminus E)$. If f is also quasianalytic in the angular variable, it follows that f vanishes outside the open disk of radius $d(0, \bar{D} \setminus \text{co}(\partial D \setminus E))$ centered at the origin. The fact that f also vanishes inside this disk, as shown above, is nontrivial. We also remark that we have not employed the support theorem in our proof of theorem 3.

Proof of corollary 4. Take any two dimensional subspace F of $\mathbb{R}^n$ such that it meets E . As noted in the proof of corollary 2 at the end of section 2.3, one ends up with the two dimensional broken ray transform in the disk $\bar{B}^n \cap F$. Since $E \cap F$ is open in the circle $S^{n-1} \cap F$, theorem 3 guarantees that $\mathcal{G}f$ restricted to those γ that lie in F uniquely determine f in $\bar{B}^n \cap F$. Since $\mathbb{R}^n$ can be written as a union of such subspaces F , $\mathcal{G}f$ determines f everywhere. $\square$

⁴ This conclusion may also be drawn as follows. Write f as a complex Fourier series: $f(r, \vartheta) = \sum_{k \in \mathbb{Z}} a_k(r) e^{ik\vartheta}$. Define $h : \mathbb{R} \rightarrow \mathbb{C}$ by $h(\varphi) = \tilde{\mathcal{G}}_{\Re_\varphi} f(\gamma) = \sum_{k \in \mathbb{Z}} e^{ik\varphi} S_{|k|}(\gamma) \mathcal{A}_{|k|} a_k(z_\gamma)$. The sequences $(S_k(\gamma) \mathcal{A}_k a_{\pm k}(r))_{k=0}^\infty$ are quasianalytic sequences, whence h is a quasianalytic function. By part (3) of lemma 17 h vanishes on some open interval, and so by quasianalyticity it vanishes everywhere. Thus each Fourier coefficient $S_{|k|}(\gamma) \mathcal{A}_{|k|} a_k(z_\gamma)$ is zero.

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