X-RAY TOMOGRAPHY OF ONE-FORMS WITH PARTIAL DATA

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ABSTRACT. If the integrals of a one-form over all lines meeting a small open set vanish and the form is closed in this set, then the one-form is exact in the whole Euclidean space. We obtain a unique continuation result for the normal operator of the X-ray transform of one-forms, and this leads to one of our two proofs of the partial data result. Our proofs apply to compactly supported covector-valued distributions.

1. INTRODUCTION

Let f be a one-form on $\mathbb{R}^n$ where $n \geq 2$. We define the X-ray transform (also known as the Doppler transform in this case) of f by the formula

$$X_1 f(\gamma) = \int_\gamma f \tag{1}$$

where γ is a line in $\mathbb{R}^n$. We freely identify one-forms with vector fields, so the differential of a scalar field corresponds to its gradient. We are interested in the problem of reconstructing f from $X_1 f$. One-forms of the form $f = \mathrm{d}\phi$ where ϕ goes to zero at infinity are always in the kernel of X_1. Thus one can only try to recover the solenoidal part f^s of the solenoidal decomposition $f = f^s + \mathrm{d}\phi$ from the data $X_1 f$. The transform X_1 is known to be solenoidally injective [28, 36], i.e. $X_1 f = 0$ implies $f = \mathrm{d}\phi$ for some scalar function ϕ. We study whether this implication holds in the whole space also in the case where we know $X_1 f$ only for a subset of lines.

We consider the following partial data problem for X_1. Let $V \subset \mathbb{R}^n$ be a nonempty open set. Assume that we know $\mathrm{d}f|_V$ and $X_1 f$ on all lines intersecting V, where $\mathrm{d}f$ is the exterior derivative or the curl of the one-form f. Can we determine the solenoidal part f^s – find f modulo potential fields – from this data? We will study the uniqueness of the partial data problem: If $\mathrm{d}f|_V = 0$ and $X_1 f = 0$ on all lines intersecting V, does it follow that $f^s = 0$?

The partial data problem for X_1 can be reduced to the following unique continuation problem for the normal operator $N_1 = X_1^* X_1$: if $\mathrm{d}f|_V = 0$ and

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$N_1 f|_V = 0$, does it follow that $f^{\mathrm{s}} = 0$? We prove that such unique continuation property holds for compactly supported covector-valued distributions under the weaker assumption that $N_1 f$ vanishes to infinite order at some point in V. The unique continuation of the normal operator implies uniqueness for the partial data problem: The solenoidal part of a one-form f is uniquely determined whenever one knows the curl of the one-form in V and the integrals of f over all lines intersecting V.

For scalar fields the uniqueness of a corresponding partial data problem and the unique continuation of the normal operator were proved in [15]. We generalize the results to one-forms using the results for scalar fields in our proofs. We also obtain partial data results and unique continuation results for the generalized X-ray transform of one-forms $X_A = X_1 \circ A$ where A is a smooth invertible matrix-valued function. As a special case of this transform we study the transverse ray transform in $\mathbb{R}^2$.

We give two alternative proofs for the partial data results. The first one uses the unique continuation of the normal operator while the second one works directly at the level of the X-ray transform and is based on Stokes' theorem.

The X-ray transform of one-forms or vector fields has applications in the determination of velocity fields of moving fluids using acoustic travel time measurements [26] or Doppler backscattering measurements [27]. Medical applications include ultrasound imaging of blood flows [17, 18, 38]. The transverse ray transform of one-forms has applications in the temperature measurements of flames [2, 34]. For two-tensors the applications include also diffraction tomography [21], photoelasticity [11] and polarization tomography [36]. For a more comprehensive treatment see the reviews [32, 33, 37] and the references therein.

We will give our main results in section 1.1 and discuss related results in section 1.2. The preliminaries are covered in section 2 and finally the theorems are proven in section 3.

1.1. **Main results.**

Here we give the main results of this paper. The proofs can be found in section 3. First we briefly go through our notation; for more detailed definitions see section 2.

Let $\mathcal{E}'(\mathbb{R}^n)$ be the space of compactly supported distributions. By $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ we mean that $f = (f_1, \ldots, f_n)$ where $f_i \in \mathcal{E}'(\mathbb{R}^n)$ for all $i = 1, \ldots, n$. We call $(\mathcal{E}'(\mathbb{R}^n))^n$ the space of compactly supported covector-valued distributions. We denote by X_1 the X-ray transform of one-forms and by $N_1 = X_1^* X_1$ its normal operator; see (19) for an explicit formula.

We say that $N_1 f$ vanishes to infinite order at $x_0 \in \mathbb{R}^n$ if it is smooth in a neighborhood of x_0 and $\partial^\beta (N_1 f)_i(x_0) = 0$ for all $\beta \in \mathbb{N}^n$ and $i = 1, \ldots, n$. We denote the exterior derivative of differential forms by d. When acting on scalars, it corresponds to the gradient.

Our first result is a unique continuation property for the normal operator N_1. The corresponding result for scalar fields and the operator $N_0 = X_0^* X_0$ of the scalar X-ray transform X_0 (see equation (12)) was proven in [15, Theorem 1.1].

Theorem 1.1. *Let $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ and $V \subset \mathbb{R}^n$ some nonempty open set. If $\mathrm{d}f|_V = 0$ and $N_1 f$ vanishes to infinite order at $x_0 \in V$, then $f = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$.*

We point out that as $\mathrm{d}f$ vanishes in V, the distribution $N_1 f$ is smooth in V by lemma 3.3 and the vanishing condition at a point is well-defined.

Theorem 1.1 is also true under the weaker assumption that $\mathrm{d}f|_V = 0$ and $\mathrm{d}(N_1 f)$ vanishes to infinite order at x_0 (see the proof in section 3.1). The condition that f is closed in V (i.e. $\mathrm{d}f|_V = 0$) is satisfied if, for example, $f|_V = 0$. When f is solenoidal (i.e. $\mathrm{div}(f) = 0$), theorem 1.1 gives the following unique continuation property: if $f|_V = N_1 f|_V = 0$, then $f = 0$.

The next result is stated directly at the level of the X-ray transform. The corresponding problem with full data was solved in [36, Theorem 2.5.1].

Theorem 1.2. *Let $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ and $V \subset \mathbb{R}^n$ some nonempty open set. Assume that $\mathrm{d}f|_V = 0$. Then $X_1 f$ vanishes on all lines intersecting V if and only if $f = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$.*

Remark 1.3. *In theorems 1.1 and 1.2 the support of the potential ϕ is contained in the convex hull of $\mathrm{spt}(f)$.*

One can view theorems 1.1 and 1.2 in terms of the global solenoidal decomposition $f = f^{\mathrm{s}} + \mathrm{d}\phi$ (see section 2.1 and equation (4)). The conclusion $f = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$ is equivalent to $f^{\mathrm{s}} = 0$.

From theorem 1.2 we obtain the following local partial data result in a bounded domain $\Omega \subset \mathbb{R}^n$. The X-ray transform of $f \in (L^2(\Omega))^n$ is defined to be $X_1 f := X_1 \widetilde{f}$ where $\widetilde{f}$ is the zero extension of f to $\mathbb{R}^n$.

Theorem 1.4. *Let $f \in (L^2(\Omega)))^n$ where $\Omega \subset \mathbb{R}^n$ is a bounded and smooth convex domain and let $V \subset \Omega$ be some nonempty open set. Assume that $\mathrm{d}f|_V = 0$. Then $X_1 f = 0$ on all lines intersecting V if and only if $f = \mathrm{d}\phi$ for some $\phi \in H_0^1(\Omega)$.*

In terms of the local solenoidal decomposition $f = f_{\Omega}^{\mathrm{s}} + \mathrm{d}\phi_{\Omega}$ (see section 2.1 and equation (5)) the conclusion $f = \mathrm{d}\phi$ for some $\phi \in H_0^1(\Omega)$ is equivalent to that $f_{\Omega}^{\mathrm{s}} = 0$.

From theorem 1.1 we also obtain the following unique continuation and partial data results for the transform $X_A = X_1 \circ A$ where $A = A(x)$ is smooth invertible matrix field. We denote by $N_A = A^T \circ N_1 \circ A$ the normal operator of X_A. When B is the constant matrix field $B(v_1, v_2) = (v_2, -v_1)$ where $(v_1, v_2) \in \mathbb{R}^2$ we write $X_B = X_\perp$ and call $X_\perp$ the transverse ray transform.

Corollary 1.5. *Let $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ and $V \subset \mathbb{R}^n$ some nonempty open set. If $\mathrm{d}(Af)|_V = 0$ and $N_A f|_V = 0$, then $f = A^{-1}(\mathrm{d}\psi)$ for some $\psi \in \mathcal{E}'(\mathbb{R}^n)$.*

Corollary 1.6. *Let $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ and $V \subset \mathbb{R}^n$ some nonempty open set. Assume that $\mathrm{d}(Af)|_V = 0$. Then $X_A f$ vanishes on all lines intersecting V if and only if $f = A^{-1}(\mathrm{d}\psi)$ for some $\psi \in \mathcal{E}'(\mathbb{R}^n)$.*

In corollaries 1.5 and 1.6 the distribution $\psi \in \mathcal{E}'(\mathbb{R}^n)$ is the potential part of the solenoidal decomposition of $Af \in (\mathcal{E}'(\mathbb{R}^n))^n$ and $\mathrm{spt}(\psi)$ is contained in the convex hull of $\mathrm{spt}(f)$. As a special case of the transform X_A we obtain

the next partial data result for the transverse ray transform $X_\perp$ which is similar to the full data result in [2].

Corollary 1.7. *Let $f \in (\mathcal{E}'(\mathbb{R}^2))^2$ and $V \subset \mathbb{R}^2$ some nonempty open set. Assume that $\mathrm{div}(f)|_V = 0$. Then $X_\perp f$ vanishes on all lines intersecting V if and only if $\mathrm{div}(f) = 0$.*

In particular, if $\mathrm{d}f|_V = \mathrm{div}(f)|_V = 0$ and both $X_1 f$ and $X_\perp f$ vanish on all lines intersecting V, then $f = 0$.

Alternatively, one can conclude in the first claim of corollary 1.7 that $f = \mathrm{curl}(\psi)$ for some $\psi \in \mathcal{E}'(\mathbb{R}^n)$ where $\mathrm{curl}(\psi) = (\partial_2 \psi, -\partial_1 \psi)$. In terms of the global solenoidal decomposition this is equivalent to that $f = f^{\mathrm{s}}$. Also in the latter claim it is enough to know the partial data of $X_1 f$ for $V \subset \mathbb{R}^2$ and the partial data of $X_\perp f$ for $W \subset \mathbb{R}^2$ where V and W can be disjoint.

Remark 1.8. *Some of the results above can be slightly generalized. Using the same proof as in theorem 1.4 one can show that corollaries 1.6 and 1.7 hold also in the local case when $f \in (L^2(B))^n$. Also in corollary 1.5 one can replace the condition $N_A f|_V = 0$ with the requirement that $N_A f$ vanishes to infinite order at $x_0 \in V$ when A is a constant matrix field. Especially, this holds for the normal operator of the transverse ray transform. One can also see from theorem 1.2 and corollary 1.7 that the X-ray transform and the transverse ray transform provide complementary information about the one-form in $\mathbb{R}^2$.*

We note that if $A = A(x)$ is not invertible for all $x \in \mathbb{R}^n$, we can still conclude in corollary 1.6 that $Af = \mathrm{d}\psi$ for some potential $\psi \in \mathcal{E}'(\mathbb{R}^n)$. Thus we obtain the "pointwise projection" Af modulo potentials from the local data for $X_A f$. We also remark that in all of our results which consider the X-ray transform in $\mathbb{R}^n$ we could replace the assumption of compact support with rapid decay at infinity. If all the derivatives of the matrix field $A = A_{ij}(x)$ grow at most polynomially, then the results are true for one-forms which components are Schwartz functions. This follows since the corresponding partial data result for scalar fields holds for Schwartz functions [15] and our method of proof is based on reducing the problem of one-forms to the problem of scalar fields.

1.2. **Related results.** Similar partial data results as in theorems 1.2 and 1.4 are previously known for scalar fields. If one knows the values of the scalar function f in an open set V, then one can uniquely reconstruct f from its local X-ray data [3, 15, 20]. In $\mathbb{R}^2$ uniqueness is also obtained under weaker assumptions: if f is piecewise constant, piecewise polynomial or analytic in V, then one can recover f uniquely from its integrals over the lines going through V [19, 20, 44]. A complementary partial data result is the Helgason support theorem [12]. According to Helgason's theorem, if $f|_C = 0$ and the integrals of f vanish on all lines not intersecting a compact and convex set C, then $f = 0$.

The normal operator of the X-ray transform of scalar fields admits a similar unique continuation property as in theorem 1.1. If f is a function which satisfies $f|_V = 0$ and $N_0 f$ vanishes to infinite order at some point in V, then $f = 0$ [15]. This is a special case of a more general unique continuation

result for Riesz potentials [15] (see equation (12)). Unique continuation of Riesz potentials is related to unique continuation of fractional Laplacians [4, 9, 15] (see also equation (13)).

Unique reconstruction of the solenoidal part of a one-form or vector field with full data is known in $\mathbb{R}^n$ [18, 26, 38, 39] and on compact simple Riemannian manifolds with boundary [14, 28]. In $\mathbb{R}^n$ uniqueness holds for compactly supported covector-valued distributions as well [36]. Some partial data results are known for one-forms. The solenoidal part can be reconstructed by knowing $X_1 f$ on all lines parallel to a finite set of planes [18, 31, 35]. When $n \geq 3$, the solenoidal part can be locally recovered near a strictly convex boundary point [40] or from the knowledge of $X_1 f$ on all lines intersecting a certain type of curve [43] (see also [8]). One can also obtain information about the singularities of the curl of a compactly supported covector-valued distribution from its X-ray data on lines intersecting a fixed curve [30]. There is a Helgason-type support theorem for one-forms: if $f|_C = 0$ and $X_1 f = 0$ on all lines not intersecting a convex and compact set C, then the solenoidal part of f vanishes [39] (see also the discussion after the alternative proof in section 3.2).

The transverse ray transform has been studied earlier with full data in $\mathbb{R}^2$ [2, 25] and also on Riemannian manifolds [16, 36] (see also [1] for a support theorem). The transverse ray transform is a special case of a more general mixed ray transform [5, 6, 36]. In higher dimensions the transverse ray transform is related to the normal Radon transform [37, 41]. In $\mathbb{R}^2$ and on certain Riemannian manifolds the knowledge of $X_1 f$ and $X_\perp f$ fully determines the one-form [2, 16]. By theorem 1.2 and corollary 1.7 this is true in $\mathbb{R}^2$ also in the case of partial data. In higher dimensions f is determined by $X_1 f$ and the normal Radon transform of f [41]. A similar transform to X_A was studied in [16, 29].

2. Preliminaries

In this section we give a brief introduction to the theory of X-ray tomography of scalar fields and one-forms in $\mathbb{R}^n$. We also define the generalized X-ray transform of one-forms. First we recall the definition and solenoidal decomposition of covector-valued distributions. We mainly follow the conventions of [7, 13, 24, 36, 39, 42] and refer the reader to them for further details.

2.1. **Covector-valued distributions and solenoidal decomposition.** We denote by $\mathcal{D}(\mathbb{R}^n)$ the space of compactly supported smooth functions, by $\mathscr{S}(\mathbb{R}^n)$ the space of rapidly decreasing smooth functions (Schwartz space) and by $\mathcal{E}(\mathbb{R}^n)$ the space of smooth functions. All spaces are equipped with their standard topologies. The spaces $\mathcal{D}'(\mathbb{R}^n)$, $\mathscr{S}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ are the corresponding topological duals. Elements of $\mathcal{D}'(\mathbb{R}^n)$ are called distributions and $\mathcal{E}'(\mathbb{R}^n)$ can be seen as the space of compactly supported distributions.

We have the continuous inclusions $\mathcal{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$. We write the dual pairing as $\langle f, \varphi \rangle$ when f is a distribution and φ is a test function.

We define the vector-valued test function space $(\mathcal{D}(\mathbb{R}^n))^n$ such that $\varphi \in (\mathcal{D}(\mathbb{R}^n))^n$ if and only if $\varphi = (\varphi_1, \ldots, \varphi_n)$ and $\varphi_i \in \mathcal{D}(\mathbb{R}^n)$ for all $i = 1, \ldots, n$. The topology of the space $(\mathcal{D}(\mathbb{R}^n))^n$ is defined as follows: a sequence φ_k converges to zero in $(\mathcal{D}(\mathbb{R}^n))^n$ if and only if $(\varphi_k)_i$ converges to zero in $\mathcal{D}(\mathbb{R}^n)$ for all $i = 1, \ldots, n$. We then define the space of covector-valued distributions $(\mathcal{D}'(\mathbb{R}^n))^n$ so that $f \in (\mathcal{D}'(\mathbb{R}^n))^n$ if and only if $f = (f_1, \ldots, f_n)$ and $f_i \in \mathcal{D}'(\mathbb{R}^n)$ for all $i = 1, \ldots, n$. The duality pairing of $f \in (\mathcal{D}'(\mathbb{R}^n))^n$ and $\varphi \in (\mathcal{D}(\mathbb{R}^n))^n$ becomes

$$(2) \qquad \langle f, \varphi \rangle = \sum_{i=1}^{n} \langle f_i, \varphi_i \rangle.$$

The spaces $(\mathcal{E}(\mathbb{R}^n))^n$, $(\mathscr{S}(\mathbb{R}^n))^n$, $(\mathcal{E}'(\mathbb{R}^n))^n$ and $(\mathscr{S}'(\mathbb{R}^n))^n$ are defined in a similar way and we call $(\mathcal{E}'(\mathbb{R}^n))^n$ the space of compactly supported covector-valued distributions. Covector-valued distributions are a special case of currents which are continuous linear functionals in the space of differential forms [7, Section III]. The components of the exterior derivative or the curl of a one-form or covector-valued distribution are

$$(3) \qquad (\mathrm{d}f)_{ij} = \partial_i f_j - \partial_j f_i.$$

One can split certain covector-valued distributions into a divergence-free part and a potential part. If $f \in (\mathcal{E}'(\mathbb{R}^n))^n$, then we have the unique decomposition [36]

$$(4) \qquad f = f^{\mathrm{s}} + \mathrm{d}\phi, \quad \mathrm{div}(f^{\mathrm{s}}) = 0$$

where $\phi \in \mathscr{S}'(\mathbb{R}^n)$ and $f^{\mathrm{s}} \in (\mathscr{S}'(\mathbb{R}^n))^n$ are smooth outside $\mathrm{spt}(f)$ and go to zero at infinity. Here ϕ is defined so that it solves the equation $\Delta\phi = \mathrm{div}(f)$ in the sense of distributions and $f^{\mathrm{s}} = f - \mathrm{d}\phi$. The decomposition (4) is known as solenoidal decomposition or Helmholtz decomposition and it holds also for $f \in (\mathscr{S}(\mathbb{R}^n))^n$ [36]. We call f solenoidal if $\mathrm{div}(f) = 0$. For the decomposition (4) this means that $f = f^{\mathrm{s}}$.

If f is supported in a fixed set, we can do the decomposition locally in that set. If $\Omega \subset \mathbb{R}^n$ is a regular enough bounded domain and $f \in (L^2(\Omega))^n$, we let ϕ_Ω to be the unique weak solution to the Poisson equation

$$(5) \qquad \begin{cases} \Delta\phi = \mathrm{div}(f) \text{ in } \Omega \\ \phi \in H_0^1(\Omega). \end{cases}$$

Then we have $f = f_\Omega^{\mathrm{s}} + \mathrm{d}\phi_\Omega$ where $f_\Omega^{\mathrm{s}} = f - \mathrm{d}\phi_\Omega \in (L^2(\Omega))^n$ and $\mathrm{div}(f_\Omega^{\mathrm{s}}) = 0$. If $f \in (C^{1,\alpha}(\overline{\Omega}))^n$ for some $0 < \alpha < 1$, then there is unique classical solution $\phi_\Omega \in C^{2,\alpha}(\overline{\Omega})$ to the boundary value problem (5) and the solenoidal decomposition holds pointwise [10].

2.2. **The X-ray transform of scalar fields.** Let Γ be the set of all oriented lines in $\mathbb{R}^n$. The X-ray transform of a function f is defined as

$$(6) \qquad X_0 f(\gamma) = \int_\gamma f \mathrm{d}s, \quad \gamma \in \Gamma$$

whenever the integrals exist. The set Γ can be parameterized as

$$(7) \qquad \Gamma = \{(z, \theta) : \theta \in S^{n-1}, \ z \in \theta^\perp\}.$$

Then the X-ray transform becomes

$$(8) \qquad X_0 f(z, \theta) = \int_{\mathbb{R}} f(z + t\theta)\mathrm{d}t$$

and it is a continuous map $X_0 \colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\Gamma)$. One can define the adjoint using the formula

$$(9) \qquad X_0^* \psi(x) = \int_{S^{n-1}} \psi(x - (x \cdot \theta)\theta, \theta)\mathrm{d}\theta$$

and it follows that $X_0^* \colon \mathcal{E}(\Gamma) \to \mathcal{E}(\mathbb{R}^n)$ is continuous. By duality we can define $X_0 \colon \mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(\Gamma)$ and $X_0^* \colon \mathcal{D}'(\Gamma) \to \mathcal{D}'(\mathbb{R}^n)$ as

$$(10) \qquad \langle X_0 f, \varphi \rangle = \langle f, X_0^* \varphi \rangle$$

$$(11) \qquad \langle X_0^* g, \eta \rangle = \langle g, X_0 \eta \rangle.$$

The normal operator $N_0 = X_0^* X_0$ is useful in studying the properties of the X-ray transform since it takes functions on $\mathbb{R}^n$ to functions on $\mathbb{R}^n$. It has an expression

$$(12) \qquad N_0 f = 2(f * |\cdot|^{1-n})$$

for continuous functions f decreasing rapidly enough at infinity. By duality the formula (12) holds also for compactly supported distributions and the normal operator becomes a map $N_0 \colon \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$. One can invert f from its X-ray transform using the normal operator by

$$(13) \qquad f = c_{0,n}(-\Delta)^{1/2} N_0 f,$$

where $c_{0,n} = (2\pi \left| S^{n-2} \right|)^{-1}$ is a constant depending on the dimension and $(-\Delta)^{1/2}$ is the fractional Laplacian of order $1/2$. The inversion formula (13) holds for $f \in \mathcal{E}'(\mathbb{R}^n)$ and for continuous functions f decreasing rapidly enough at infinity.

2.3. **The X-ray transform of one-forms.** Let f be a one-form on $\mathbb{R}^n$. We define its X-ray transform as

$$(14) \qquad X_1 f(\gamma) = \int_\gamma f, \quad \gamma \in \Gamma$$

whenever the integrals exist. Using the parametrization (7) for Γ we obtain

$$(15) \qquad X_1 f(z, \theta) = \int_{\mathbb{R}} f(z + t\theta) \cdot \theta \mathrm{d}t.$$

It follows that $X_1 \colon (\mathcal{D}(\mathbb{R}^n))^n \to \mathcal{D}(\Gamma)$ is continuous. The adjoint is defined as

$$(16) \qquad (X_1^* \psi)_i(x) = \int_{S^{n-1}} \theta_i \psi(x - (x \cdot \theta)\theta, \theta)\mathrm{d}\theta$$

and $X_1^* \colon \mathcal{E}(\Gamma) \to (\mathcal{E}(\mathbb{R}^n))^n$ is also continuous. Thus we can define $X_1 \colon (\mathcal{E}'(\mathbb{R}^n))^n \to \mathcal{E}'(\Gamma)$ and $X_1^* \colon \mathcal{D}'(\Gamma) \to (\mathcal{D}'(\mathbb{R}^n))^n$ as

$$(17) \qquad \langle X_1 f, \varphi \rangle = \langle f, X_1^* \varphi \rangle$$

$$(18) \qquad \langle X_1^* g, \eta \rangle = \langle g, X_1 \eta \rangle.$$

If $f \in (L^p(\Omega))^n$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $p \geq 1$, we define its X-ray transform as $X_1 f := X_1 \widetilde{f}$ where $\widetilde{f} \in (\mathcal{E}'(\mathbb{R}^n))^n$ is the zero extension of f.

Like in the scalar case we define the normal operator $N_1 = X_1^* X_1$ and it satisfies the formula

$$(19) \qquad (N_1 f)_i = \sum_{j=1}^{n} \frac{2 x_i x_j}{|x|^{n+1}} * f_j.$$

The normal operator can be extended to a map $N_1 \colon (\mathcal{E}'(\mathbb{R}^n))^n \to (\mathcal{D}'(\mathbb{R}^n))^n$ and the formula (19) holds for $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ and also for continuous one-forms decreasing rapidly enough at infinity. One can invert the solenoidal part of f using the normal operator by

$$(20) \qquad f^{\mathrm{s}} = c_{1,n} (-\Delta)^{1/2} N_1 f,$$

where $c_{1,n} = |S^n|$ is a constant depending on the dimension and $(-\Delta)^{1/2}$ operates componentwise. The formula (20) holds for $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ and also for continuous one-forms decreasing rapidly enough at infinity.

2.4. The generalized X-ray transform of one-forms.
Let $A = A(x)$ be a smooth matrix-valued function on $\mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$ the matrix $A(x)$ is invertible. We define the transform X_A of a one-form f as

$$(21) \qquad X_A f(\gamma) = \int_{-\infty}^{\infty} A(\gamma(t)) f(\gamma(t)) \cdot \dot{\gamma}(t) \mathrm{d}t = X_1(Af)(\gamma), \quad \gamma \in \Gamma.$$

Thus X_A can be seen as the X-ray transform of the "rotated" one-form Af. The transform X_A can also be defined on compactly supported covector-valued distributions. We first let $\langle Af, \varphi \rangle = \langle f, A^T \varphi \rangle$ for $f \in (\mathcal{D}'(\mathbb{R}^n))^n$ and a test function φ where A^T is the pointwise transpose of A and $(A^T \varphi)(x) = A^T(x)\varphi(x)$. Then clearly A is a map $A \colon (\mathcal{E}'(\mathbb{R}^n))^n \to (\mathcal{E}'(\mathbb{R}^n))^n$. Therefore we can define $X_A \colon (\mathcal{E}'(\mathbb{R}^n))^n \to \mathcal{D}'(\Gamma)$ as $X_A f = X_1(Af)$. One easily sees that the adjoint is $X_A^* = A^T \circ X_1^*$ and the normal operator becomes $N_A = A^T \circ N_1 \circ A$. By the discussion above the normal operator can be extended to a map $N_A \colon (\mathcal{E}'(\mathbb{R}^n))^n \to (\mathcal{D}'(\mathbb{R}^n))^n$.

Let B be the constant matrix field on $\mathbb{R}^2$ defined as $B(v_1 e_1 + v_2 e_2) = v_2 e_1 - v_1 e_2$ where $\{e_1, e_2\}$ is any orthonormal basis of $\mathbb{R}^2$. The matrix B corresponds to a clockwise rotation by 90 degrees. We then define the transverse ray transform $X_\perp$ by letting $X_\perp = X_B$. It follows that the transverse ray transform provides complementary information about the solenoidal decomposition compared to the X-ray transform, i.e. X_1 determines the solenoidal part and $X_\perp$ determines the potential part of a one-form [2] (see also theorem 1.2 and corollary 1.7).

3. Proofs of the main results

We give two alternative proofs for the partial data results. The first proof uses the unique continuation of the normal operator and the second proof works directly at the level of the X-ray transform. Both proofs are based on the corresponding results for scalar fields.

3.1. **Proofs using the unique continuation of the normal operator.**
In this section we prove our main results using the unique continuation
property of the normal operator. We need the following lemmas in our
proofs.

Lemma 3.1 ([15, Theorem 1.1]). *Let $V \subset \mathbb{R}^n$ be some nonempty open set
and $g \in \mathcal{E}'(\mathbb{R}^n)$. If $g|_V = 0$ and $\partial^\beta (N_0 g)(x_0) = 0$ for some $x_0 \in V$ and all
$\beta \in \mathbb{N}^n$, then $g = 0$.*

Lemma 3.2 (Poincaré lemma). *Let $g \in (\mathcal{D}'(\mathbb{R}^n))^n$ such that $\mathrm{d}g = 0$. Then
there is $\eta \in \mathcal{D}'(\mathbb{R}^n)$ such that $\mathrm{d}\eta = g$. If $g \in (\mathcal{E}'(\mathbb{R}^n))^n$, then $\eta \in \mathcal{E}'(\mathbb{R}^n)$.*

The proof of lemma 3.2 can be found in [13, 22]. We first prove the
unique continuation result for the normal operator. The proof is based on
the fact that we can reduce the unique continuation problem of N_1 to a
unique continuation problem of N_0 acting on the components of $\mathrm{d}f$.

The assumptions of theorem 1.1 come in two stages. We first assume that
$\mathrm{d}f|_V = 0$. To make sense of the next assumption that $N_1 f$ vanishes at x_0
to infinite order, we need to ensure that it is smooth near this point. This
is given by the next lemma.

Lemma 3.3. *Let $V \subset \mathbb{R}^n$ be an open set and $f \in (\mathcal{E}'(\mathbb{R}^n))^n$. If $\mathrm{d}f|_V = 0$,
then $N_1 f|_V$ is smooth.*

Proof. Take any $x_0 \in V$ and a small open ball B centered at it and contained
in V. As $\mathrm{d}f|_B = 0$, the Poincaré lemma applied in the ball B (lemma 3.2
is applicable because B is diffeomorphic to $\mathbb{R}^n$) gives $f|_B = \mathrm{d}h$ for some
$h \in \mathcal{D}'(B)$. Let $B' \subset B$ be a smaller ball with the same center, and let
$\chi \in \mathcal{D}(B)$ be a bump function so that $\chi|_{B'} \equiv 1$. If we let $h' = \chi h \in \mathcal{E}'(\mathbb{R}^n)$,
then $f = \mathrm{d}h' + g$, where $g \in (\mathcal{E}'(\mathbb{R}^n))^n$ with $g|_{B'} = 0$.

As $X_1(\mathrm{d}h') = 0$ (cf. (26)), we have $N_1 f = N_1 g$. Because $g|_{B'} = 0$, it
follows from properties of convolutions that $N_1 f$ is smooth in B'. Now
that $N_1 f$ is smooth in a neighborhood of any point in V, the claim follows.
□

Proof of theorem 1.1. The normal operator has an expression

$$(22) \qquad (N_1 f)_i = \sum_{j=1}^n \frac{2 x_i x_j}{|x|^{n+1}} * f_j.$$

We can write the kernel as

$$(23) \qquad \frac{2 x_i x_j}{|x|^{n+1}} = \frac{2}{n-1} \left(\delta_{ij} |x|^{1-n} - \partial_i (x_j |x|^{1-n}) \right)$$

and we obtain

$$(24) \qquad (N_1 f)_i = \frac{2}{n-1} \left(\frac{1}{2} N_0 f_i - \sum_{j=1}^n x_j |x|^{1-n} * \partial_i f_j \right).$$

We can calculate that

$$(25) \qquad \partial_k (N_1 f)_i - \partial_i (N_1 f)_k = \frac{1}{n-1} N_0 (\partial_k f_i - \partial_i f_k).$$

This can be interpreted as $\mathrm{d}(N_1 f) = (n-1)^{-1} N_0(\mathrm{d}f)$, where the scalar nor-
mal operator N_0 acts on the 2-form $\mathrm{d}f$ componentwise to produce another

2-form. The normal operator commutes with the exterior derivative in this sense.

Since $N_1 f$ vanishes to infinite order at $x_0 \in V$ also $N_0(\partial_k f_i - \partial_i f_k)$ vanishes to infinite order at x_0. Using lemma 3.1 we obtain $\mathrm{d}f = 0$. By lemma 3.2 there is $\phi \in \mathcal{E}'(\mathbb{R}^n)$ such that $\mathrm{d}\phi = f$. This concludes the proof. $\qquad\square$

Lemma 3.1 is false if no restrictions are imposed on $g|_V$ [20, 24], and the assumption $g|_V = 0$ is the most convenient. Consequently, the assumption $\mathrm{d}f|_V = 0$ in theorem 1.1 is important. This condition is invariant under gauge transformations of the field f.

If $f^{\mathrm{s}}|_V = N_1 f|_V = 0$, then one can alternatively use the unique continuation of the fractional Laplacian $(-\Delta)^s$, $s \in (0, 1)$, to prove the unique continuation of the normal operator [9]. This follows since $(-\Delta)^{1/2} f^{\mathrm{s}} = c_{1,n}(-\Delta) N_1 f$ where $f^{\mathrm{s}} \in (H^r(\mathbb{R}^n))^n$ for some $r \in \mathbb{R}$ when $f \in (\mathcal{E}'(\mathbb{R}^n))^n$. One can also make use of the fact that $(-\Delta)^{-1/2}$ is a Riesz potential and use its unique continuation properties [15] (see equation (20)).

The rest of the results follow easily from theorem 1.1.

Proof of theorem 1.2. Let $f = \mathrm{d}\phi$ where $\phi \in \mathcal{E}'(\mathbb{R}^n)$. Then $\mathrm{d}\phi \in (\mathcal{E}'(\mathbb{R}^n))^n$ and using the definition of the X-ray transform on distributions we obtain

$$(26) \qquad \langle X_1(\mathrm{d}\phi), \varphi \rangle = \langle \mathrm{d}\phi, X_1^* \varphi \rangle = \langle \phi, \mathrm{div}(X_1^* \varphi) \rangle = 0.$$

Here we used the fact that $\mathrm{div}(X_1^* \varphi) = 0$ which follows from a straightforward computation. This proves the other direction. Assume then that $\mathrm{d}f|_V = 0$. Since $X_1 f = 0$ on all lines intersecting V we obtain $N_1 f|_V = 0$. Theorem 1.1 implies that $f = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$. This concludes the proof. $\qquad\square$

Proof of theorem 1.4. The other direction of the claim follows as in the proof of theorem 1.2 since $H_0^1(\Omega) \subset \mathcal{E}'(\mathbb{R}^n)$ in the sense of zero extension. Then assume that $\mathrm{d}f|_V = 0$ and $X_1 f = 0$ on all lines intersecting V. Let $\widetilde{f} \in (\mathcal{E}'(\mathbb{R}^n))^n$ be the zero extension of f. The assumptions imply that $\mathrm{d}\widetilde{f}|_V = 0$ and $X_1 \widetilde{f} = 0$ on all lines intersecting V. Theorem 1.2 implies that $\widetilde{f} = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^n)$. Since $\Delta\phi = \mathrm{div}(\widetilde{f}) \in H^{-1}(\mathbb{R}^n)$ we have $\phi \in H^1(\mathbb{R}^n)$ by elliptic regularity. On the other hand, $\mathrm{spt}(\phi) \subset \overline{\Omega}$ and hence $\phi \in H_0^1(\Omega)$ [23, Theorem 3.33]. The claim follows from the fact that $\mathrm{d}\phi = \widetilde{f} = f$ in Ω. $\qquad\square$

Proof of corollary 1.5. We know that the normal operator is $N_A = A^T \circ N_1 \circ A$. The assumptions imply that $\mathrm{d}(Af)|_V = N_1(Af)|_V = 0$. By theorem 1.1 we obtain that $Af = \mathrm{d}\psi$ for some $\psi \in \mathcal{E}'(\mathbb{R}^n)$. This gives the claim. $\qquad\square$

Proof of corollary 1.6. The claim follows directly from corollary 1.5 and from the fact that $X_A = X_1 \circ A$. $\qquad\square$

In theorems 1.1 and 1.2 one has $\mathrm{spt}(\phi) \subset \mathrm{Conv}(\mathrm{spt}(f))$ where $\mathrm{Conv}(\mathrm{spt}(f))$ is the convex hull of $\mathrm{spt}(f)$. This follows from the fact that ϕ has compact support and $\mathrm{d}\phi$ vanishes in the connected set $\mathrm{Conv}(\mathrm{spt}(f))^c$. This was pointed out in remark 1.3.

In corollaries 1.5 and 1.6 one also has $\mathrm{spt}(\psi) \subset \mathrm{Conv}(\mathrm{spt}(f))$. This holds since $\mathrm{d}\psi$ vanishes in the connected set $\mathrm{Conv}(\mathrm{spt}(Af))^c$ and $\mathrm{spt}(Af) = \mathrm{spt}(f)$.

Proof of corollary 1.7. Assume first that $\mathrm{div}(f) = 0$. Since f is a covector-valued distribution in $\mathbb{R}^2$ we can identify $\mathrm{d}f = \partial_1 f_2 - \partial_2 f_1$. It follows that $\mathrm{d}(Bf) = -\mathrm{div}(f) = 0$ and thus $Bf = \mathrm{d}\eta$ for some $\eta \in \mathcal{E}'(\mathbb{R}^n)$ by lemma 3.2. Therefore $X_\perp f = X_1(Bf) = X_1(\mathrm{d}\eta) = 0$. Assume then that $\mathrm{div}(f)|_V = 0$ and $X_\perp f = 0$ on all lines intersecting V. As above we obtain that $\mathrm{d}(Bf)|_V = 0$ and $X_\perp f = 0$ on all lines intersecting V. Corollary 1.6 implies that $f = B^{-1}(\mathrm{d}\psi)$ for some $\psi \in \mathcal{E}'(\mathbb{R}^n)$. From this we obtain that $\mathrm{div}(f) = 0$.

Assume then that $\mathrm{d}f|_V = \mathrm{div}(f)|_V = 0$ and both $X_1 f$ and $X_\perp f$ vanish on all lines intersecting V. By the discussion above we obtain that $\mathrm{div}(f) = 0$. On the other hand, theorem 1.2 implies that $f = \mathrm{d}\phi$ for some $\phi \in \mathcal{E}'(\mathbb{R}^2)$. Therefore $\Delta \phi = 0$ and since ϕ has compact support we must have $\phi = 0$, i.e. $f = 0$. $\qquad\square$

3.2. **Proofs based on Stokes' theorem.** In this section we give alternative proofs for the partial data results using Stokes' theorem in $\mathbb{R}^n$. A similar approach was used in [18, 38] in the case of full data. We prove the results first for compactly supported smooth one-forms and then use standard mollification argument to prove them for compactly supported covector-valued distributions. We only need to prove theorem 1.2 since the rest of the partial data results follow from it. We will use the following lemma.

Lemma 3.4 ([15, Theorem 1.2])**.** *Let $V \subset \mathbb{R}^n$ be some nonempty open set and $g \in \mathcal{E}'(\mathbb{R}^n)$. If $g|_V = 0$ and $X_0 g = 0$ on all lines intersecting V, then $g = 0$.*

Alternative proof of theorem 1.2. By lemma 3.2 it suffices to show that $\mathrm{d}f = 0$. Assume first that $n = 2$ and $f \in (\mathcal{D}(\mathbb{R}^2))^2$. Let γ be any (oriented) line going through V and ν the counterclockwise rotated normal to γ. We denote by $\gamma_h = h\nu + \bar{\gamma}$ the reversed parallel line shifted by $h > 0$ in the direction of ν so that γ_h also intersects V. By assumption $\int_\gamma f = \int_{\gamma_h} f = 0$.

We form a closed loop $\widetilde{\gamma}_h$ enclosing counterclockwise a rectangular region R_h such that the ends are outside $\mathrm{spt}(f)$ (see figure 1). When considered as chains, we have $\widetilde{\gamma}_h = \partial R_h$. As the chains $\gamma - \gamma_h$ and $\widetilde{\gamma}_h$ differ only outside the support of f, the integrals coincide. By Stokes' theorem

$$(27) \qquad 0 = \int_\gamma f - \int_{\gamma_h} f = \int_{\widetilde{\gamma}_h} f = \int_{\partial R_h} f = \int_{R_h} \mathrm{d}f = \int_{R_h} \star \mathrm{d}f \, \mathrm{d}\mu,$$

where $\star$ is the Hodge star and μ is the 2-Hausdorff measure.

We aim to show that the scalar function $\star \mathrm{d}f$ vanishes. Scaling with h, we find

$$(28) \qquad 0 = \lim_{h \to 0} \frac{1}{h} \int_{R_h} \star \mathrm{d}f \, \mathrm{d}\mu = \int_\gamma \star \mathrm{d}f \, \mathrm{d}s.$$

Now that $\star \mathrm{d}f|_V = 0$ and $X_0(\star \mathrm{d}f)(\gamma) = 0$ for all lines γ meeting V, lemma 3.4 implies that $\star \mathrm{d}f = 0$ and thus also $\mathrm{d}f = 0$ in the whole plane.

Consider then the case $n \geq 3$ for a compactly supported smooth one-form f. Let $P \subset \mathbb{R}^n$ be any two-plane meeting V and $\iota_P \colon P \to \mathbb{R}^n$ the corresponding inclusion. By the argument above for the two-form $\iota_P^* f$ in the plane P we have that $\iota_P^* \mathrm{d}f = 0$ for all such planes.

Take any point $z \in \mathbb{R}^n$. For any plane P through z that intersects V we have $\iota_P^* \mathrm{d}f = 0$. This is an open subset of the Grassmannian of 2-planes through z, so $\mathrm{d}f(z) = 0$. As the point z was arbitrary, we have $\mathrm{d}f = 0$.

Finally, let $f \in (\mathcal{E}'(\mathbb{R}^n))^n$ and define $f_\epsilon = f * j_\epsilon = (f_1 * j_\epsilon, \ldots, f_n * j_\epsilon)$ where $j_\epsilon \in \mathcal{D}(\mathbb{R}^n)$ is the standard mollifier. Then $f_\epsilon \in (\mathcal{D}(\mathbb{R}^n))^n$ and $\langle X_1(f * j_\epsilon), \varphi \rangle = \langle X_1 f, X_0 j_\epsilon \circledast \varphi \rangle$ where

$$(29) \qquad (h \circledast g)(z, \theta) = \int_{\theta^\perp} h(z - y, \theta) g(y, \theta) \mathrm{d}y.$$

Hence there is a nonempty open set $W \subset V$ such that for small $\epsilon > 0$ we have $f_\epsilon|_W = 0$ and $X_1 f_\epsilon = 0$ on all lines intersecting W. Using the above reasoning for smooth one-forms we obtain $0 = \mathrm{d}f_\epsilon = \mathrm{d}f * j_\epsilon$ for small $\epsilon > 0$. Taking $\epsilon \to 0$ we get $\mathrm{d}f = 0$. $\qquad \square$
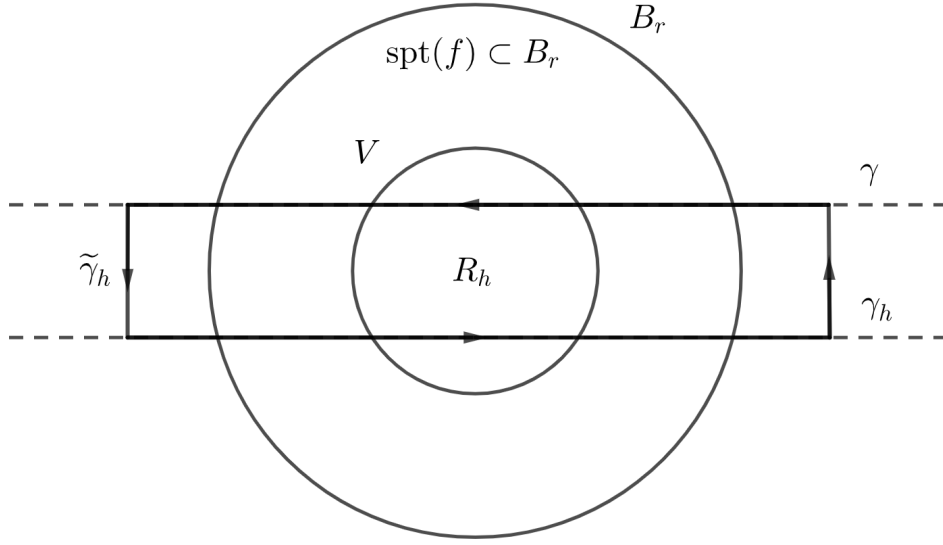

FIGURE 1. Basic idea of the alternative proof of theorem 1.4 when $n = 2$. We may assume that f is supported in a ball B_r. We form a closed loop $\widetilde{\gamma}_h$ with the lines γ and γ_h (dashed) enclosing the rectangular region R_h. Then we apply Stokes' theorem and a limit argument $h \to 0$ together with a known partial data result for scalar fields to obtain that $\mathrm{d}f = 0$.

Now the proof of theorem 1.4 follows in the same way from theorem 1.2 as before using the zero extension $\widetilde{f}$. Corollaries 1.6 and 1.7 are also direct consequences of theorem 1.2 since $X_A = X_1 \circ A$. Moreover, the above alternative proof can be used to prove a complementary support theorem for the transform X_A: if $\mathrm{d}(Af)|_C = 0$ and $X_A f = 0$ on all lines not intersecting a convex and compact set C, then $f = A^{-1}(\mathrm{d}\psi)$ for some potential ψ. See [39] for a similar support theorem for the X-ray transform X_1.

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