FYSA231 Quantum Mechanics I, part B Fall 2011

Exercise 9 Return to the exercise box in FYS1 lobby before 4 pm on Fri, Nov 25

1. (a) Starting from the Schrödinger equation for the relative motion (p. 217), derive the radial Schrödinger equation (p. 219) by separating the variables r, θ and ϕ. [Use the relation between ∇^2 and $\hat{\mathbf{L}}^2$ and that $\hat{\mathbf{L}}^2 Y_{lm}(\theta,\phi) = \hbar^2 l(l+1) Y_{lm}(\theta,\phi)$.] Use your radial Schrödinger equation to derive the Schrödinger equation form (p. 223) analogous to the one-dimensional case.

   (b) In the case of the infinitely deep potential well on pp. 221–222 we found out that the radial wavefunctions were spherical Bessel functions $R(r) = Aj_l(kr)$, where $k^2 = 2mE/\hbar^2$. Let's consider the case $l = 1$ more closely. Show that the energy eigenvalues can be found by solving the transcendental equation $\tan ka = ka$. Sketch a graphical solution and check the order of magnitude of your solutions by comparing them to the picture on p. 222. Finally, show that the eigenenergies corresponding to large values of n in the $l = 1$ case are $E_{n\gg 1} \approx \frac{\hbar^2\pi^2}{2ma^2}(n+\frac{1}{2})^2$.

2. Consider a *finite* three-dimensional spherically symmetric potential well.

$$V(r) = \begin{cases} -V_0, & \text{kun}\, r \leq a; \\ 0, & \text{kun}\, r \geq a. \end{cases}$$

   Show that the energies of bound states in the case $l = 0$ can be calculated from the transcendental equation

$$-\cot \kappa a = \sqrt{\frac{2mV_0 a^2/\hbar^2}{(\kappa a)^2} - 1}.$$

   Sketch a graphical solution and show that the system does not have any bound states if $V_0 a^2 < \frac{\pi^2\hbar^2}{8m}$.

   [Tips: It is a good idea to use the p. 223 form of the radial Schrödinger equation. By doing this, your solution $u(r)$ is exponentially damped when $r > a$ and sinusoidal when $r < a$ (and you don't need to worry about spherical Bessel or Neuman functions). By requiring continuity of the wave function and its derivative, you obtain the required equations which you can use to derive the transcendental equation just like in the one-dimensional case.]

Turn the page!

The purpose of the following exercises is to practice using CG-coefficients. Find the required CG-coefficients from the table given in the lecture notes.

3. Consider the coupling of two angular momenta into a total angular momentum, $\hat{\mathbf{J}} = \hat{\mathbf{J}}_\mathbf{1} + \hat{\mathbf{J}}_\mathbf{2}$ and its quantum numbers as in the lecture notes. Form all the states $|j_1, j_2, j, m\rangle_c$ which correspond to the largest value of the total angular momentum quantum number j with the help of the uncoupled basis $|j_1, j_2, m_1, m_2\rangle_u$ when $j_1 = 1$ and $j_2 = \frac{1}{2}$.

4. Let's consider the coupling of the *orbital angular momentum* and the spin angular momentum $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$ in the case of electron (spin-$\frac{1}{2}$ particle). Let the electron state in the uncoupled basis be $|l = 1, s = \frac{1}{2}, m_l = 0, m_s = \frac{1}{2}\rangle_u$.

   Let's measure the z component of the total angular momentum $\hat{J}_z$ and the magnitude of the total angular momentum $\hat{\mathbf{J}}^2$ from this state simultaneously. What combinations of the quantum numbers m and j can be obtained and with what probabilities?

5. (a) Derive the boxed result found in the lower part of p. 246 (calculate the matrix element in two different ways).

   (b) Let's continue practicing the use of the CG-table (p. 243). Consider two angular momenta $j_1 = 2$ and $j_2 = \frac{3}{2}$. [This can be, for an example, the coupling of the orbital angular momentum (now $j_1 = l = 2$) and spin ($j_2 = s = \frac{3}{2}$) of the spin-$\frac{3}{2}$-particle into the total angular momentum.]

      (i) Express the coupled state $|j_1 = 2, j_2 = \frac{3}{2}, j = \frac{5}{2}, m = \frac{1}{2}\rangle_c$ in the uncoupled basis.

      (ii) Express the uncoupled state $|j_1 = 2, j_2 = \frac{3}{2}, m_1 = 1, m_2 = -\frac{1}{2}\rangle_u$ in the coupled basis.

6. (2 bonus points) [This is a physically important problem, but it has been left optional because it is mathematically laborous.] Consider the bound states of the Hydrogen atom as we did in the lectures.

   (a) Transform the radial Schrödinger equation (p. 226) into the dimensionless form (see the boxes in the middle of p. 226), and with the trial function $u(r) = \rho^{l+1} e^{-\rho} w(\rho)$ derive the differential equation found on p. 226 for the function $w(\rho)$.

   (b) Substitute the series solution $w(\rho) = \sum_k a_k \rho^k$ into this differential equation and derive the recursion formula for the coefficients a_k found on p. 227 and make sure you understand why the series needs to have finite number of terms and where the energy eigenvalues, aka Bohr's formula on p. 228, come from.

7. (0.5 bonus points) Has something been left unclear so far in the course? Ask something about the course content or related to the course, so that it will be clarified more in the lectures, the exercise sessions, or the course home page.