FYSA232 Quantum Mechanics I, part B Fall 2011

Exercise 6 Return to the exercise box in FYS1 lobby before 4 pm on Fri, Nov 4

1. (2 p) Derive the boxed WKB-result for the transmission coefficient (tunneling probability) for an α-particle tunneling through the one-dimensional Coulomb potential barrier $V_C(r)$ (see $V_C(r)$ on p. 149). First, calculate the integral

$$\int_R^b dr \sqrt{\frac{2m}{\hbar^2}[V_C(r) - E_\alpha]}$$

analytically exactly. For this apply a suitable change of variables. After this, consider the limit $b \gg R$ and show that

$$T \approx \exp[-4\frac{Z_1}{\sqrt{E_\alpha/\mathrm{MeV}}}].$$

You'll probably need the series expansions $\arccos x = \frac{\pi}{2} - x + \mathcal{O}(x^3)$ and $\sqrt{1-x} = 1 + \mathcal{O}(x)$. Other hints: For α-particle (= 2 protons + 2 neutrons) $mc^2 \approx 940$ MeV and charge number $Z_\alpha = 2$. The fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$. (Notice that you won't need the numerical values for the physical constants c, ϵ_0 and e because these are absorbed in the fine structure constant α and in the mass energy mc^2).

2. (4 p) Using the definition for the operators $\hat{a}$ and $\hat{a}^\dagger$ found on p.152, show that

   a) $[\hat{a}, \hat{a}^\dagger] = 1$

   b) $\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$, where $\hat{N} = \hat{a}^\dagger\hat{a}$

   c) $[\hat{N}, \hat{a}] = -\hat{a}$

   d) $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$

   e) $\hat{N}\hat{a}^\dagger |n\rangle = (n+1)\hat{a}^\dagger |n\rangle$, where $\hat{N}|n\rangle = n|n\rangle$

   f) $\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$

   g) $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

   (Tip: Use the result $|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n |0\rangle$ for f) and g).)

3. (2 p) Starting from the ground state wave function found on p. 159 $\psi_0(x) = \langle x|n=0\rangle$ and applying the ladder operator $\hat{a}^\dagger$ to the ground state $|0\rangle$, derive the wave function of the first excited state $\langle x|1\rangle = \psi_1(x)$ (i.e. apply the ideas on pp. 159–160 without using the Hermite polynomials).

4. (2 p) Use the ladder operators $\hat{a}^\dagger$ and $\hat{a}$ and the Dirac's notation to derive the results (p. 161) for the standard deviations $(\Delta x)_n$ and $(\Delta p)_n$ computed from an energy eigenstate $|n\rangle$.

5. (1 bonus point) Has something been left unclear so far in the course? Ask something about the course content or related to the course, so that it will be clarified more in the lectures, the exercise sessions, or the course home page.