Exercise 5 Return to the exercise box in FYS1 lobby before 4 pm on Fri, Oct 14

1. (2 p) Let's consider the time evolution of states and the calculation of probabilities for the measurement values of observables. Consider a two-state system with orthonormal state vectors $|1\rangle$ and $|2\rangle$. The Hamilton operator of the system acts as follows:

$$\begin{aligned} \hat{H}\,|1\rangle &= E_1\,|1\rangle\,, \\ \hat{H}\,|2\rangle &= E_2\,|2\rangle\,. \end{aligned}$$

   Let there also be an observable A and the corresponding operator $\hat{A}$ has the properties

$$\begin{aligned} \hat{A}\,|1\rangle &= +i\hbar\,|2\rangle\,, \\ \hat{A}\,|2\rangle &= -i\hbar\,|1\rangle\,. \end{aligned}$$

   Let the initial state of the system at $t = 0$ be $|\psi(0)\rangle = \frac{3}{5}\,|1\rangle + \frac{4}{5}\,|2\rangle$. Assuming that the system is left undisturbed until the time t, and the observable A is measured then, which are the possible values for A and with what probability? Is A a constant of motion? (*A comment:* Something similar to this problem would make a good problem for the exam because it requires the understanding of the basic concepts and calculation techniques.)

2. (a) (1 p) By using the results derived in the lecture notes, show in a one-dimensional case that if $\hat{p}\,|\psi(t)\rangle = |\phi(t)\rangle$, then in the x-representation $-i\hbar\frac{\partial}{\partial x}\psi(t,x) = \phi(t,x)$.

   (b) (1 p) Respectively, show that if $\hat{x}\,|\psi(t)\rangle = |\phi(t)\rangle$, then we have in the p-representation $+i\hbar\frac{\partial}{\partial p}\psi(t,p) = \phi(t,p)$.

   (c) (1 p) Using **x** and **p** representations, show that if $\langle\psi|\psi\rangle = 1$, then $\int d^3\mathbf{x}|\psi(\mathbf{x})|^2 = \int d^3\mathbf{p}|\psi(\mathbf{p})|^2 = 1$. This result is known as *the Parseval formula* in mathematics.

3. (2 p) Show that the Fourier transform of the mininum wave packet in the **x** representation found on p. 117 is the result $\psi(p)$ found on p. 118. With as short a calculation as possible (use our previous results), show that for the normal distribution $\rho(x)$ on p. 117 the expectation value of x is $\langle x\rangle$ and the standard deviation is b.

4. (3 p) Consider the following model for H_2^+ molecule-ion where the electron moves in the double-well potential formed by δ-functions

$$V(x) = -\Omega[\delta(x-a) + \delta(x+a)],$$

   where $\Omega > 0$. Find the energy spectrum of the bound states when

   (a) $a = \frac{\hbar^2}{m\Omega}$. Notice the symmetry of the potential $V(-x) = V(x)$ and arrange your solutions in even and odd stationary wave functions. The energy eigenvalues are determined by transcendental equations, solve them numerically or present a graphical solution from which you can estimate the required numerical values.

   (b) $a = \frac{1}{3}\frac{\hbar^2}{m\Omega}$. How does the situation differ qualitatively from above?

5. (Bonus problem worth 2 points) Starting from the stationary Schrödinger equation and using the potential barrier found on p. 135, derive the results for the reflection and the transmission coefficients R and T found on p. 136.

6. (Bonus problem worth 2 points) Prove the δ-function properties a–g on p. 102.

7. (0 points) Has something been left unclear so far in the course?