FYSA231 Quantum Mechanics I, part A Fall 2011

Exercise 4 Return to the exercise box in FYS1 lobby before 4 pm on Fri, Oct 7

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1. Consider a three-state system with three states with the following properties

$$\begin{aligned} \hat{H}|1\rangle &= \hbar\omega|1\rangle & \hat{A}|1\rangle &= \lambda|2\rangle \\ \hat{H}|2\rangle &= 2\hbar\omega|2\rangle & \hat{A}|2\rangle &= \lambda|1\rangle \\ \hat{H}|3\rangle &= 3\hbar\omega|3\rangle & \hat{A}|3\rangle &= 2\lambda|3\rangle, \end{aligned}$$

where $\hat{H}$ is the Hamilton operator, $\hat{A}$ corresponds to an another observable, and $\{|1\rangle, |2\rangle, |3\rangle\}$ is one of the system's orthonormal basis. Write the matrix representations of $\hat{H}$ and $\hat{A}$ and find the eigenvalues and eigenvectors of these matrix representations in the eigenbasis of $\hat{H}$. What is the matrix representation of $\hat{H}$ in the eigenbasis of $\hat{A}$?

2. Prove the following identities for operators $\hat{A}$, $\hat{B}$ and $\hat{C}$:
   (a) $\left[\hat{A}+\hat{B},\hat{C}\right] = \left[\hat{A},\hat{C}\right] + \left[\hat{B},\hat{C}\right]$
   (b) $\left[\hat{A}\hat{B},\hat{C}\right] = \hat{A}\left[\hat{B},\hat{C}\right] + \left[\hat{A},\hat{C}\right]\hat{B}$
   (c) $\left[\hat{A},\hat{B}\right]^\dagger = \left[\hat{B}^\dagger,\hat{A}^\dagger\right]$
   (d) $\left[\hat{A},\left[\hat{B},\hat{C}\right]\right] + \left[\hat{B},\left[\hat{C},\hat{A}\right]\right] + \left[\hat{C},\left[\hat{A},\hat{B}\right]\right] = 0$

3. (a) Show that the matrix representation of a hermitian operator $\hat{A}$ is hermitian.
   (b) Consider hermitian operators $\hat{A}$ and $\hat{B}$ and use the definitions (lecture notes, p. 84)

$$\hat{A}' = \hat{A} - \langle A\rangle \qquad \hat{B}' = \hat{B} - \langle B\rangle \qquad \hat{M} = \hat{A}'\hat{B}' + \hat{B}'\hat{A}' \qquad \left[\hat{A},\hat{B}\right] = i\hat{C},$$

   to show that $\left[\hat{A}',\hat{B}'\right] = i\hat{C}$, and that $\hat{M}$ and $\hat{C}$ are hermitian.

4. Starting from the general result $\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\left\langle\left[\hat{H},\hat{A}\right]\right\rangle$ for a time-independent hermitian operator $\hat{A}$, prove the *Ehrenfest theorem*,

$$\frac{d\langle x\rangle}{dt} = \left\langle\frac{p}{m}\right\rangle \qquad \text{and} \qquad \frac{d\langle p\rangle}{dt} = -\left\langle\frac{dV}{dx}\right\rangle,$$

for a nonrelativistic one-particle Hamilton operator $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$.

5. Let's examine the energy-time-uncertainty principle (p. 97–98) in the case of a harmonic oscillator. Let the wave function of the system at $t = 0$ be a linear combination

$$\psi(t=0,x) = a\psi_0(x) + b\psi_1(x), \qquad a > 0,\ b > 0,\ a^2+b^2 = 1$$

where wave functions ψ_0 and ψ_1 correspond to the two lowest energy eigenstates (see p. 55). Calculate ΔE, $\frac{d\langle x\rangle}{dt}$ and Δx, and using them find δt. How much is $\Delta E|\delta t|$? This is quite a lengthy exercise – but you'll get an extra point if you can show that $\Delta E|\delta t| \geq \hbar/2$.

6. Has something been left unclear so far in the course? Ask something about the course content or related to the course, so that it will be clarified more in the lectures, the exercise sessions, or the course home page. (The question gives one bonus point.)