A BRIEF INTRODUCTION TO THE ATTENUATED X-RAY TRANSFORM AND SPECT

JESSE RAILO

Abstract. We define the attenuated X-ray transform and have a discussion on its injectivity. We descibe the medical imaging menthod SPECT (Single-photon emission computed tomography), which is the most common application of the attenuated X-ray transform. This note is for 20 minutes student talk for the course "Analysis and X-ray tomography" at the University of Jyväskylä in Autumn 2017. We do not consider any mathematical details due to lack of time, e.g. how to justify interchange from the integral geometry problem to the dynamical problem or vice versa.

1. The attenuated X-ray transform

Let (M, g) be a compact two-dimensional Riemannian manifold with strictly convex boundary. In this talk we simply restric to the case where $M := \overline{B(0,1)} \subset \mathbb{R}^2$ is the closed ball with the Euclidean metric. Let us denote by ν_x the unit outer normal of M at $x \in \partial M$.

Let $a \in C^\infty(M)$ be the attenuation coefficient in the body M (known/unkown real function). Let $f \in C^\infty(M)$ be a radiation source in the body M (unkown real function). Let $\gamma_{x,v}$, $(x, v) \in SM$, denote the line spanned by $x + tv, t \in \mathbb{R}$. Let $\tau(x, v)$ denote the espace time for $\gamma_{x,v}$, i.e. the time $\gamma_{x,v}$ hits to the boundary. The attenuated ray transform is defined by

$$I_a f(x, v) := \int_0^{\tau(x,v)} f(\gamma_{x,v}(t)) \exp\left(\int_0^t a(\gamma_{x,v}(s))ds\right) dt$$

for any $(x, v) \in \partial_+ SM$, i.e. $x \in \partial M$ and $\nu_x \cdot v \leq 0$.

There are three fundamental injectivity problems related to this transformation:

1) If a and $I_a f$ is known, can we find f?
2) If f and $I_a f$ is known, can we find a?
3) If $I_a f$ is known, can we find both a and f?

The first one is well studied and understood in many cases including the Euclidean geometry, hyperbolic geometry, simple surfaces and manifolds that admit a smooth strictly convex foliation: the answer is positive in those geometries. Usually one assumes things to be smooth.

In the Euclidean space however there might be lower regularity results as well. The third question is more difficult, but has also practical interest due SPECT (and I do not know what can be said about it). I have not given much thought to the second question either.

We also remark that the X-ray transform is a special case of zero attenuation. One also easily notices that I_a is linear mapping for fixed attenuations. One important fact is also that the weight never vanish.

Example 1.1. If $a \equiv -1$, then $I_a f(x,v) = \int_0^{\tau(x,v)} f(\gamma_{x,v}(t)) \exp(-t) dt$.

In 80s and 90s the exponential X-ray transform was also considered, it may refer to this type of transforms.

2. SPECT

Single-photon emission computed tomography (SPECT) is a nuclear medicine tomographic imaging technique using gamma rays. The technique requires delivery of a gamma-emitting radioisotope (a radionuclide) into the patient, normally through injection into the bloodstream (Wikipedia, 3.10.2017). One then measures the outgoing radiation and hope to locate the sources under the condition that there is no incoming radiation (sources outside the body).

Let $u(x,v)$ be the density of particles at x travelling in the direction of v with speed $|v| = 1$. The dynamical system behind the attenuated ray transform (and SPECT) is

$$v \cdot \nabla u(x,v) + a(x) u(x,v) = f(x)$$

in SM. The boundary conditions are

$$u|_{\partial_- SM} = 0, \quad I_a f = u|_{\partial_+ SM}$$

where the former is *no incoming radiation* condition and the latter is the out going radiation, i.e. the data which can be measured in SPECT. It is possible to solve the outgoing radiation (the direct problem) using the dynamical model and the no incoming radiation condition, and it is exactly $I_a f$.

We remark that the attenuation is the same attenuation as the unkown function in CT, therefore in practical situation one needs to find it before using CT (or to have a good estimate for it somehow). Here the physical role of f is different than in CT, it represents radiation sources inside the body and we want to locate them. However, in CT (as we have seen in the course) the dynamical system only differs to the one in attenuated X-ray transform by changing the operator X to $X + a$.

One possible way to study the question is using the integrating factors $Xw = a$. Then $e^{-w}Xe^{w} = X + a$. Then this problem reduces back to the form $Xu = g$. Now however g depends also both the base point and the direction in comparison to the dynamical system approach in CT where f only depends on the base point. Therefore this leads further technicalities if one wants to show injectivity by this method, but it can be done and the method works also in simple manifolds. One can see e.g. the paper by Salo and Uhlmann in 2011 solving this problem. I can also recommend a survey by Finch in 2003, which gives many of the details in the Euclidean case.

Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD) FI-40014 University of Jyväskylä, Finland

E-mail address: `jesse.t.railo@jyu.fi`