Talk for X-Ray course: "Riesz potentiales"

Giovanni Covi

September 28, 2017

In this talk we want to prove the following theorem:

Theorem 6.1 : *The Riesz potential* $I_\alpha : C_c(\mathbb{R}^n) \to C(\mathbb{R}^n)$ *defined by* $I_\alpha f = c_\alpha^{-1} f * |x|^{\alpha-n} = c_\alpha^{-1} \int_{\mathbb{R}^n} \frac{f(x-y)}{|y|^{n-\alpha}} dy$ *is an injection* $\forall \alpha \in (0, n)$.

Let us briefly recall the definition of Fourier transform, which will be studied in somewhat deeper detail in a future talk:

Definition 1 : *Let* $f : \mathbb{R}^n \to \mathbb{C}$ *be a function such that* $f \in L^1(\mathbb{R}^n)$. *Then its Fourier transform is* $\hat{f}(\xi) = \mathcal{F}f(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\,x\cdot\xi} f(x)dx$. *Its inverse Fourier transform is* $\check{f}(x) = \mathcal{F}^{-1}f(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\,x\cdot\xi} f(\xi)d\xi$. *Both* $\hat{f}$ *and* $\check{f}$ *are functions* $\mathbb{R}^n \to \mathbb{C}$.

The reason why f is taken in the space $L^1(\mathbb{R}^n)$ is clear: in this way we know that its Fourier transform is well defined, because

$$|\hat{f}(\xi)| = c(n) \left| \int_{\mathbb{R}^n} e^{-i\,x\cdot\xi} f(x)dx \right| \leq c(n) \int_{\mathbb{R}^n} |f(x)|dx < \infty$$

In the proof of the theorem, we would like to take the Fourier transform of the functions $I_\alpha f(x)$ and $|x|^{\alpha-n}$, but there is no reason why this would be legitimate by Definition 1. In fact in general $I_\alpha f(x) \in C(\mathbb{R}^n)$ is not integrable, while we know for sure that $\int_{\mathbb{R}^n} |x|^{\alpha-n} dx = \infty$, because it is heavy at infinity. This is why we need a way to extend the definition to larger function spaces.

It can be proved that if $v, w \in L^1(\mathbb{R}^n)$, then the following formula holds:

$$\int_{\mathbb{R}^n} \hat{v}(x)w(x)dx = \int_{\mathbb{R}^n} v(\xi)\hat{w}(\xi)d\xi$$

This is a property of the Fourier transform, and so in order to prove it we need $v, w \in L^1(\mathbb{R}^n)$; but if we supposedly start from the formula itself, there is actually no need for such regularity: v and w could belong to other function spaces, and the formula would still make sense as long as $\hat{v}(x)w(x), v(\xi)\hat{w}(\xi) \in L^1(\mathbb{R}^n)$. Let us define the two following function spaces:

Definition 2 : *The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consists of all $\phi \in C^\infty(\mathbb{R}^n)$ such that $\sup_{x\in\mathbb{R}^n} |x^\alpha \partial_\beta \phi(x)| < \infty$, $\forall \alpha, \beta \in \mathbb{Z}^n_+$. An element of $\mathcal{S}(\mathbb{R}^n)$ is also called a "rapidly decreasing function".*

Definition 3 : *The space of tempered distributions $\mathcal{T}(\mathbb{R}^n)$ is the dual space of $\mathcal{S}(\mathbb{R}^n)$. An element of $\mathcal{T}(\mathbb{R}^n)$ is also called a "slowly growing function".*

Some examples of Schwartz functions are all compactly supported functions, and also $e^{-t|\xi|^2}, \forall t > 0$; for the use we will make of them, a tempered distribution can be considered as a function from $\mathbb{R}^n$ to $\mathbb{C}$ such that its integral against any Schwartz function is finite. Let us state some remarks:

- $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$, $\forall p \in [1, \infty]$. In particular, every Schwartz function has a finite integral, and we can thus consider its Fourier transform.
- The Fourier transform is a linear isomorphism $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. In particular, the Fourier transform of a Schwartz function is still a Schwartz function.
- The function $|x|^{\alpha-n}$ is a tempered distribution. In fact, if ϕ is any Schwartz function, the integral of f against ϕ can be divided into two parts: one in a ball B close to the origin, where the integral is finite because ϕ is bounded in B and $f \in L^1(B)$, and one on the rest of $\mathbb{R}^n$, where the integral is also finite since ϕ would integrate finitely against any slowly increasing function, and f even decreases.
- For similar reasons, it is easily shown that $I_\alpha f \in \mathcal{T}(\mathbb{R}^n)$ (the computations are made easier by the use of the change of variable $x - y = z$).

We are now ready to give a new definition of Fourier transform:

Definition 4 : *Let $f \in \mathcal{T}(\mathbb{R}^n)$. If $g : \mathbb{R}^n \to \mathbb{C}$ is such that the following formula*

$$\int_{\mathbb{R}^n} \hat{\phi}(x) f(x) dx = \int_{\mathbb{R}^n} \phi(\xi) g(\xi) d\xi$$

holds $\forall \phi \in S(\mathbb{R}^n)$, then g is called the (weak) Fourier transform of f (in the sense of tempered distributions), and we write $\hat{f} = g$.

It is clear from Definition 4 that the Fourier transform of a tempered distribution is still a tempered distribution. According to this definition, we now want to show as a preliminary result that $\mathcal{F}(|x|^{\alpha-n})(\xi) = b_\alpha |\xi|^{-\alpha}$. For the sake of simplicity, before that we will carry out a computation:

Lemma 1 : *We have $\mathcal{F}\left[e^{-t|\xi|^2}\right] = c(n)\, t^{-n/2} e^{-|x|^2/4t}$, for all $x, \xi \in \mathbb{R}^n,\ t > 0$.*

Proof : Firts of all observe that $\forall t > 0,\ e^{-t|\xi|^2}$ belongs to $L^1(\mathbb{R}^n)$, so we can apply the normal definition of Fourier transform without fear. Let us compute

$$\mathcal{F}\left[e^{-t|\xi|^2}\right] = \int_{\mathbb{R}^n} e^{-ix\cdot\xi - t|\xi|^2} d\xi = \prod_{j=1}^{n} \int_{\mathbb{R}} e^{-ix_j\cdot\xi_j - t\xi_j^2} d\xi_j$$

By means of Fubini's theorem, we have divided the integral over $\mathbb{R}^n$ in n integrals of $\mathbb{R}$; of course we could use Fubini, because the integrability of the absolute value of the function follows from what we said in the beginning. Let us now change variable: $\xi_j = t^{-1/2}z_j - ix_jt^{-1}/2$

$$\begin{aligned}
(...) &= \prod_{j=1}^{n} \int_{\mathbb{R}} e^{-ix_j\cdot(t^{-1/2}z_j - ix_jt^{-1}/2) - t(t^{-1/2}z_j - ix_jt^{-1}/2)^2} t^{-1/2} dz_j \\
&= \prod_{j=1}^{n} \int_{\gamma} e^{-x_j^2/4t - z_j^2} dz_j = e^{-|x|^2/4t} \prod_{j=1}^{n} \int_{\mathbb{R}} e^{-z_j^2} dz_j \\
&= e^{-|x|^2/4t} \prod_{j=1}^{n} \left(\frac{\pi}{t}\right)^{1/2} = c(n)\, t^{-n/2} e^{-|x|^2/4t}
\end{aligned}$$

In the course of these computations, γ is the line in the complex plane resulting from the change of variable; later it gets deformed back into $\mathbb{R}$. $\square$

Lemma 2 : *Let $\alpha \in (0, n)$. Then $\mathcal{F}(|x|^{\alpha-n})(\xi) = b_\alpha|\xi|^{-\alpha}$, where the Fourier transform is intended in the sense of tempered distributions.*

Proof : By definition, we want to prove that $\forall \phi \in \mathcal{S}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \hat{\phi}(x)|x|^{\alpha-n} dx = b_\alpha \int_{\mathbb{R}^n} \phi(\xi)|\xi|^{-\alpha} d\xi$$

In order to do so, we first of all compute that by using the change of variable $s = t|\xi|^2$ in the definition of the Γ function,

$$\Gamma(z+1) := \int_0^\infty s^z e^{-s} ds = |\xi|^{2+2z} \int_0^\infty t^z e^{-t|\xi|^2} dt$$

so that, if we take $z = (\alpha - 2)/2$ such that $z > -1$, we are left with

$$|\xi|^{-\alpha} = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{(\alpha-2)/2} e^{-t|\xi|^2} dt$$

Now we can use this to compute

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi(\xi)|\xi|^{-\alpha} d\xi &= \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}^n} \phi(\xi) \int_0^\infty t^{(\alpha-2)/2} e^{-t|\xi|^2} dt d\xi \\
&= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{(\alpha-2)/2} \int_{\mathbb{R}^n} \phi(\xi) e^{-t|\xi|^2} d\xi dt
\end{aligned}$$

where Fubini is legitimate by the fact that the integrand is in L^1 by definition of Schwartz function. We now use Parseval's identity, which in short is $\int uv = \int \hat{u}\hat{v}$; this will be likely shown in the talk about Fourier transforms. So we get

$$\begin{aligned}(...) &= b_\alpha \int_0^\infty t^{(\alpha-2)/2} \int_{\mathbb{R}^n} \hat{\phi}(x)\mathcal{F}\left[e^{-t|\xi|^2}\right] dxdt \\ &= b_{\alpha,n} \int_0^\infty t^{(\alpha-2)/2} \int_{\mathbb{R}^n} \hat{\phi}(x) t^{-n/2} e^{-|x|^2/4t} dxdt \\ &= b_{\alpha,n} \int_{\mathbb{R}^n} \hat{\phi}(x) \int_0^\infty t^{(\alpha-n-2)/2} e^{-|x|^2/4t} dtdx\end{aligned}$$

The last passage is legitimate again by the same Fubini - Schwartz argument, while in the first one we applied Lemma 1. Now we change variable $t = r|x|^2$, so that we get

$$\begin{aligned}(...) &= b_{\alpha,n} \int_{\mathbb{R}^n} \hat{\phi}(x) \int_0^\infty r^{(\alpha-n-2)/2} |x|^{\alpha-n-2} e^{-1/4r} dr |x|^2 dx \\ &= b_{\alpha,n} \int_{\mathbb{R}^n} |x|^{\alpha-n} \hat{\phi}(x) \int_0^\infty r^{(\alpha-n-2)/2} e^{-1/4r} drdx \\ &=: b_{\alpha,n} \int_{\mathbb{R}^n} |x|^{\alpha-n} \hat{\phi}(x) J dx\end{aligned}$$

The proof will thus be complete as soon as we will have shown that $J := \int_0^\infty r^{(\alpha-n-2)/2} e^{-1/4r} dr$ is a (finite) constant. We start by observing that since $\alpha \in (0,n)$, then $\beta := (\alpha - n - 2)/2 \in (-n/2 - 1, -1) \subset (-n, -1)$. So if we write

$$J = \int_0^1 r^\beta e^{-1/4r} dr + \int_1^\infty r^\beta e^{-1/4r} dr =: J_1 + J_2$$

we immediately conclude $J_2 \leq \int_1^\infty r^\beta dr < \infty$. For J_1 we observe that the integrand function is continuous in $(0,1)$ (in fact its limit in 0 is 0), therefore it is bounded and so $J_2 < \infty$. Hence we have recovered our thesis. □

Now we are ready to prove the theorem.

Proof of Theorem 6.1 : As we always do, in order to prove injectivity we will show that $I_\alpha f = 0 \Rightarrow f = 0$, $\forall \alpha \in (0,n)$. This makes sense because the Riesz potential is defined as an integral, and as such it is a linear operator. Let us apply the Fourier transform:

$$\mathcal{F}(I_\alpha f) = \mathcal{F}(c_\alpha^{-1} f * |x|^{\alpha-n}) = c_\alpha^{-1} \mathcal{F}(f) \mathcal{F}(|x|^{\alpha-n}) = b_{\alpha,n} \mathcal{F}(f) |\xi|^{-\alpha}$$

Here we have used the basic property of Fourier transforms $\mathcal{F}(a*b) = \mathcal{F}(a)\mathcal{F}(b)$, which changes a convolution to a standard product. Again, this will be shown in a future talk. In the last passage we have applied our Lemma 2. Now, if we know that $I_\alpha f = 0$, of course its Fourier transform will also be 0; this implies that $b_{\alpha,n}\mathcal{F}(f)|\xi|^{-\alpha} = 0$ and eventually that $\mathcal{F}(f) = 0$, since the other terms are strictly positive. The result $f = 0$ is then a consequence of the injectivity of the Fourier transform. □