Theorem 11.1. The Fourier transform is a bijection $\mathcal{F}: L^2(\mathbb{R}^2) \to \cancel{\mathbb{R}} L^2(\mathbb{R}^2)$, given by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(\xi)\, d\xi, \qquad (1)$$

~~then~~ for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and extend by continuity to the rest of $L^2(\mathbb{R}^n)$.

The inverse Fourier transform $\mathcal{F}^{-1}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is given by

$$(\mathcal{F}^{-1}f)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi\cdot x} f(\xi)\, d\xi.$$

the Fourier transform is unitary in the sense than

$$\int_{\mathbb{R}^n} \overline{g(x)} f(x)\, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} \overline{\mathcal{F}g(\xi)}\, \mathcal{F}f(\xi)\, d\xi.$$

Definition: The Schwartz space $\mathcal{S}(\mathbb{R}^n)$, or the space of rapidly decreasing functions, is the set of infinitely differentiable complex functions on $\mathbb{R}^n$ for which the seminorms

$$\|\varphi\|_{\alpha,\beta} = \|x^\alpha \partial^\beta \varphi(x)\|_{L^\infty(\mathbb{R}^n)}$$

are finite for all $\alpha, \beta \in \mathbb{N}^n$. Equivalently, $\mathcal{S}(\mathbb{R}^n)$ is the space is the space of functions for which the norms

$$\|\varphi\|_N = \sum_{|\beta| \le N} \| \langle x \rangle^N \partial^\beta \varphi \|_{L^\infty(\mathbb{R}^n)}$$

$$\langle x \rangle = (1+|x|^2)^{\frac{1}{2}}$$

are ~~fin~~ finite for all $N \in \mathbb{N}$.

Remarks ~~Example~~: 1. $C_0^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$. $\exp(-|x|^2) \in \mathscr{S}(\mathbb{R}^n)$, $\exp(-|x|) \notin \mathscr{S}(\mathbb{R}^n)$

2. $\mathscr{S}(\mathbb{R}^n)$ is a complete metric space.

3. $\mathscr{S}(\mathbb{R}^n)$ is a dense subspace of $L^p(\mathbb{R}^n)$ if $1 \le p < \infty$.

~~Let~~

~~$C_0(\mathbb{R}^n) = \{f:$~~

Definition: The Fourier transform of a function $f \in \mathscr{S}(\mathbb{R}^n)$ is the function $\mathcal{F}(f)$: $\mathbb{R}^n \to \mathbb{C}$ defined by

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx, \qquad \xi \in \mathbb{R}^n.$$

Then inverse Fourier transform of $f \in \mathscr{S}(\mathbb{R}^n)$ is defined by

$$\mathcal{F}^{-1}(f)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\, d\xi, \quad x \in \mathbb{R}^n.$$

Theorem (Fourier transform on Schwartz space). The Fourier transform is an isomorphism from $\mathscr{S}(\mathbb{R}^n)$ onto $\mathscr{S}(\mathbb{R}^n)$. The inverse map is the inverse Fourier transform: one has $\mathcal{F}^{-1}(\mathcal{F}(f)) = \mathcal{F}(\mathcal{F}^{-1}(f)) = f$ for

$f \in \mathcal{S}(\mathbb{R}^n)$.

Lemma B: For any $f \in \mathcal{S}(\mathbb{R}^n)$, the Fourier transform $\mathcal{F}(f)$ is a C^∞ function from $\mathbb{R}^n$ to $\mathbb{C}$ and $\partial^\alpha \mathcal{F}(f) \in L^\infty(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}^n$.

Proof. The function $\mathcal{F}(f)$ is bounded since

$$|\mathcal{F}(f)(\xi)| \leq \int_{\mathbb{R}^n} |e^{-ix\cdot\xi} f(x)|\, dx = \|f\|_{L^1}. \qquad (1)$$

By ~~and~~ definition

$$\frac{\mathcal{F}(f)(\xi + he_k) - \mathcal{F}(f)(\xi)}{h} = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, \frac{e^{-ihx_k} - 1}{h}\, dx.$$

The estimate

$$\left| \frac{e^{-ihx_k} - 1}{h} \right| = \left| \int_0^{x_k} e^{-iht}\, dt \right| \leq |x_k|.$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$, we have $\left| |x_k| f(x) \right| < \infty$, then by dominated convergence theorem, we have

$$\frac{\partial}{\partial \xi_k} \mathcal{F}(f)(\xi) = \mathcal{F}((-ix_k) f(x)) \qquad (*)$$

It follows that the first partial derivatives of $\mathcal{F}(f)$ are bounded continuous functions. Since $x^\alpha f(x)$ is in $\mathcal{S}(\mathbb{R}^n)$ for any multi-index α, we may repeat the process to see that derivatives of any order are bounded continuous functions in $\mathbb{R}^n$.

Theorem (Properties of Fourier transform) Let $f \in \mathcal{S}(\mathbb{R}^n)$, $x_0, \xi_0 \in \mathbb{R}^n$, $c > 0$ and $\alpha, \beta \in \mathbb{N}^n$. Then the following identities hold:

(1) $\mathcal{F}(\tau_{x_0} f(x)) = e^{-ix_0\cdot\xi}\hat{f}(\xi)$ (translation)

(2) $\mathcal{F}(e^{ix\cdot\xi_0} f(x)) = \tau_{\xi_0}\hat{f}(\xi)$ (modulation)

(3) $\mathcal{F}(f(cx)) = c^{-n}\hat{f}(\frac{\xi}{c})$ (Scaling)

(4) $\mathcal{F}(D^\alpha f(x)) = \xi^\alpha \hat{f}(\xi)$ (derivative)

(5) $\mathcal{F}((-x)^\beta f(x)) = D^\beta \hat{f}(\xi)$ (Polynomial)

Proof.

$$\mathcal{F}\left(\frac{\partial}{\partial_k} f(x)\right) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\frac{\partial f}{\partial x_k}(x)\,dx = -\int_{\mathbb{R}^n}\frac{\partial}{\partial x_k}(e^{-ix\cdot\xi})f(x)\,dx$$

$$= (i\xi_k)\int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx = (i\xi_k)\hat{f}(\xi).$$

Thus $\mathcal{F}(D_{x_k} f) = \xi_k \hat{f}$, and (4) follows by ~~ite~~ iteration.

(5) follow iteration of $(*)$.

Lemma. $\mathcal{F}$ and $\mathcal{F}^{-1}$ map $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ continuously.

Proof. Let $f \in \mathcal{S}$ and let α, β ~~be~~ be multi-indices.

Then ~~$\#f$~~ $\hat{f} \in C^\infty(\mathbb{R}^n)$ and $\|\hat{f}\|_{L^\infty} \le \|f\|_{L^1}$.

Then by (4) and (5) in last Theorem

$$\|\hat{f}\|_{\alpha,\beta} = \sup_{\xi\in\mathbb{R}^n} |\xi^\alpha D^\beta \hat{f}(\xi)| = \sup_{\xi\in\mathbb{R}^n} |(i\xi)^\alpha D^\beta \hat{f}(\xi)|.$$

$$= \sup_{\xi\in\mathbb{R}^n} |(i\xi)^\alpha \mathcal{F}((-ix)^\beta f(\xi))|.$$

$$= \sup_{\xi\in\mathbb{R}^n} |\mathcal{F}(D^\alpha((-ix)^\beta f(\xi)))|.$$

Since $D^\alpha(-ix)^\beta f(\xi) \in \mathcal{S}(\mathbb{R}^n)$ So, $\|\hat{f}\|_{\alpha,\beta} < \infty$ and $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$

Fourier transform is linear, and by (3-1) we have

$$\|\hat{f}\|_{\alpha,\beta} \leq \|D^\alpha((-ix)^\beta f(x))\|_{L^1}.$$

Leibniz rule. we have

$$D^\alpha[(-ix)^\beta f(x)] = \sum_{k=1}^m C_k x^{\alpha_k} D^{\beta_k} f(x)$$ for some constants C_k and multi-indices α_k, β_k. So

$$\|\hat{f}\|_{\alpha,\beta} \leq C\sum_{k=1}^m \|x^{\alpha_k} D^{\alpha_k} f\|_{L^1} \leq C\sum_{k=1}^m \|x^{\alpha_k+n+1} D^{\beta_k} f\|_{L^\infty}.$$

Now if $f_j \to 0$ in $\mathcal{S}$ we have $\|\hat{f_j}\|_{\alpha,\beta} \to 0$ for all α, β, showing that $\mathcal{F}$ is continuous. The proof that $\mathcal{F}^{-1}$ is continuous is similar.

Lemma. The function $\phi_n \in \mathcal{S}(\mathbb{R}^n)$ given by

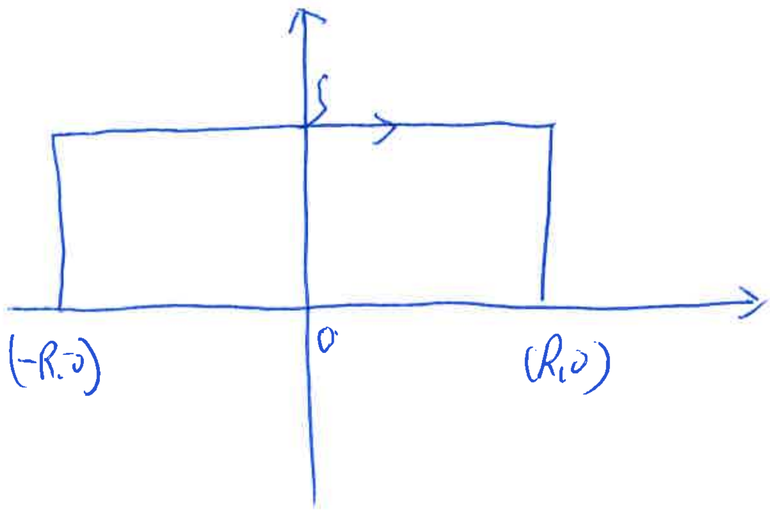
$$\phi_n(x) = e^{-\frac{1}{2}|x|^2}$$

satisfies $\hat{\phi}_n = (2\pi)^{\frac{n}{2}} \phi_n$ and $\phi_n(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\phi}_n(x)\, dx$.

Proof.

$$\hat{\phi}_1(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} e^{-\frac{1}{2}x^2}\, dx = e^{-\frac{1}{2}\xi^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\xi)^2}\, dx.$$

Integrating $e^{-\frac{1}{2}z^2}$ along the rectangular contour with corners at $(\pm R, 0)$ and $(\pm R, \xi)$ gives



$$\int_{-R}^{R} e^{-\frac{1}{2}(x+i\xi)^2}\, dx = \int_{-R}^{R} e^{-\frac{1}{2}x^2} + \int_0^{\xi} \left(e^{-\frac{1}{2}(R+iy)^2} - e^{-\frac{1}{2}(-R+iy)^2} dy \right) dy.$$

$\downarrow R \to \infty$ $\qquad$ $\downarrow R \to \infty$

$\sqrt{2\pi}$ $\qquad$ 0.

So $\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\xi)^2}\, dx = \sqrt{2\pi}$.

So

$$\hat{\phi}_1(\xi) = \sqrt{2\pi}.$$

and
$$\phi_n(x) = \phi_1(x_1) \cdots \phi_1(x_n)$$

$$\hat{\phi}_n(\xi) = \hat{\phi}_1(\xi_1) \cdots \hat{\phi}_1(\xi_n) = (2\pi)^{\frac{n}{2}} \phi_n.$$

Theorem (Fourier inversion theorem) For any $f \in \mathscr{S}(\mathbb{R}^n)$ one has the inversion formula $\mathcal{F}^{-1}\mathcal{F} f = f$, that is,

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n}$$

Proof. For $f, g \in \mathscr{S}(\mathbb{R}^n)$, Use Fubini's theorem, we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y} f(x) g(y)\, dx dy = \int_{\mathbb{R}^n} \hat{f}(y) g(y)\, dy$$

$$\|$$

$$\int_{\mathbb{R}^n} f(x) \hat{g}(x)\, dx$$

Now lets us choose $g(x) = \phi_n(\frac{x}{\varepsilon})$ $\cancel{g = \phi_n}$ (Defined in the Lemma above).

we have

$$\int_{\mathbb{R}^n} \hat{f}(y)\,\phi_n\left(\frac{y}{c}\right)dy = \int_{\mathbb{R}^n} f(x)\,\hat{\phi}_n\left(\frac{x}{c}\right)dx$$

~~&~~ Scaling Property of ~~F~~ F

$$= \int_{\mathbb{R}^n} f(x)\, c^n \hat{\phi}_n(\cancel{x c}\, cx)\,dx$$

$$= \int_{\mathbb{R}^n} \cancel{f(\tfrac{x}{c})\hat{\phi}_n}\; f\left(\frac{x}{c}\right)\hat{\phi}_n(x)\,dx$$

Let $c \to \infty$. we have

$$\phi(0)\int_{\mathbb{R}^n}\hat{f}(y) = f(0)\int_{\mathbb{R}^n}\hat{\phi}_n(x)\,dx \quad \text{Then}$$

$$f(0) = (2\pi)^{-n}\int_{\mathbb{R}^n}\hat{f}(x)\,dx = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{2i0\cdot x}\hat{f}(x)\,dx$$

for general $y \in \mathbb{R}^n$, just use the translation property of the Fourier ~~#~~ transform.

$\square$

Definition: Let $\mathcal{S}'(\mathbb{R}^n)$ be the set of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$. Thus

$$\mathcal{S}'(\mathbb{R}^n) = \{T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C};\ T \text{ linear and } T(\varphi_j) \to 0 \text{ whenever } \varphi_j \to 0 \text{ in } \mathcal{S}(\mathbb{R}^n)\}$$

The elements of $\mathcal{S}'$ are called tempered distributions.

Definition. The Fourier transform of any tempered distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ is the tempered distribution $\hat{T} = \mathcal{F}T$ defined by

$$\hat{T}(\varphi) = T(\hat{\varphi}).$$

Similarly, the inverse Fourier transform of $T \in \mathcal{S}'$ is the distribution $\check{T} = \mathcal{F}^{-1}T$ for which $\check{T}(\varphi) = T(\check{\varphi})$.

Theorem (Fourier transform on tempered distributions). The Fourier transform is a ~~bijection~~ bijective map from $\mathcal{S}'(\mathbb{R}^n)$ onto $\mathcal{S}'(\mathbb{R}^n)$. It is continuous in the sense that

$$T_j \to T \text{ in } \mathcal{S}' \Rightarrow \hat{T}_j \to \hat{T} \text{ in } \mathcal{S}'.$$

One has the inversion formula

$$\mathcal{F}^{-1}\mathcal{F}T = \mathcal{F}\mathcal{F}^{-1}T = T, \qquad T \in \mathcal{S}'.$$

Let

$$C_0(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ continuous and } f(x) \to 0 \text{ as } |x| \to \infty\}.$$

We equip $C_0(\mathbb{R}^n)$ with the $L^\infty(\mathbb{R}^n)$ norm, and then ~~$C_0(\mathbb{R}$~~ $C_0(\mathbb{R}^n)$ is a Banach space

Theorem (Riemann-Lebesgue) The Fourier transform is a continuous map from $L^1(\mathbb{R}^n)$ into $C_0(\mathbb{R}^n)$. For any $f \in L^1$ the Fourier transform is given by the usual formula

$$(3) \qquad \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx, \qquad \xi \in \mathbb{R}^n$$

Proof. If $f \in \mathcal{S}$ then we already know that $\hat{f} \in C_0(\mathbb{R}^n)$ and $\|\hat{f}\|_{L^\infty} \le \|f\|_{L^1}$. This means that $\mathcal{F} : \mathcal{S} \subset L^1 \to C_0$ is a bounded linear map from a dense subspace of L^1 to C_0, hence has a ~~bounded~~ unique bounded extension $\Phi : L^1 \to C_0$ with $\|\Phi(f)\|_{L^\infty} \le \|f\|_{L^1}$.

We wish to show that $\Phi = \mathcal{F}|_{L^1}$ where $\mathcal{F}$ is the Fourier transform on $\mathcal{S}'$. For this we take any $f \in L^1$ and choose a sequence $(f_j) \subset \mathcal{S}$ such that $f_j \to f$ in L^1. Then $\mathcal{F}f_j \to \Phi(f)$ in ~~L^∞ C_0,~~ L^∞. hence also in $\mathcal{S}'$. but also $\mathcal{F}f_j \to \mathcal{F}f$ in $\mathcal{S}'$ by the Theorem above. Since limits in $\mathcal{S}'$ are unique, we have $\Phi(f) = \mathcal{F}(f)$ as distributions. The formula (3) is given by

$$\Phi(f)(\xi) = \lim_{j\to\infty} \hat{f}_j(\xi) = \lim_{j\to\infty} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f_j(x)\, dx = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

where the last equality follows since ~~$\|f_j \to f$~~ $\|f_j - f\|_{L^1} \to 0$. //

Theorem (Plancherel) The Fourier transform is an isomorphism from $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$. It is isometric in the sense that

$$\|\hat{f}\|_{L^2(\mathbb{R}^n)} = (2\pi)^{\frac{n}{2}} \|f\|_{L^2(\mathbb{R}^n)}.$$

The transform is given by

$$(**) \quad \hat{f}(\xi) = \underset{R\to\infty}{\mathrm{l.i.m}} \int_{|x|\le R} e^{-ix\cdot\xi} f(x)\, dx$$

where l.i.m means that the limit is in L^2.

Proof. If $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$ and $\|\hat{f}\|_{L^2} = (2\pi)^{\frac{n}{2}} \|f\|_{L^2}$ by Parseval's identity. Thus $\mathcal{F} : \mathcal{S} \subset L^2 \to L^2$ is an isometry from a dense subspace of L^2 to L^2 and extends uniquely into an isometry $\Phi : L^2 \to L^2$.

It follows from Schwarz's inequality and a similar argument as in the proof of the preceding theorem that Φ and $\mathcal{F}|_{L^2}$ coincide. For $(**)$ let $B_R = B(0,R)$ and let χ_{B_R} be the ~~charac~~ characteristic function. Then for any $f \in L^2$ we have $\chi_{B_R} f \to f$ in L^2 as $R \to \infty$, thus $\widehat{(\chi_{B_R} f)} \to \hat{f}$ in L^2 by what we have already proved. This gives

$$\hat{f}(\xi) = \underset{R\to\infty}{\mathrm{l.i.m}}\ (\widehat{\chi_{B_R} f})(\xi) = \underset{R\to\infty}{\mathrm{l.i.m}} \int_{|x|\le R} e^{-ix\cdot\xi} f(x)\, dx,$$

the last equation coming from the preceding theorem since $\chi_{B_R} f$ is in L^1.

17.

Remarks:

Parseval's identity:

$$\int_{\mathbb{R}^n} \hat{g}(\xi) f(\xi)\, d\xi = (2\pi)^n \int_{\mathbb{R}^n} \overline{g(x)} f(x)\, dx.$$