# 1 About Fourier series

In this presentation we prove theorem 2.3 of the lectures notes [1]. To prove it we need some basic tools and results from functional analysis. Because we are considering locally square-integrable $2\pi$-periodic functions, we can restrict our attention to the cube $Q = [-\pi, \pi]^n$. Define inner product on $Q$ as

$$\langle f, g \rangle = \frac{1}{(2\pi)^n} \int_Q f(x)\overline{g(x)} \mathrm{d}x,$$

when $f, g \in L^2(Q)$, so that the norm becomes $\|f\| = \sqrt{\langle f, f \rangle}$.

Let $e_k : Q \to \mathbb{C}$, $e_k(x) = e^{ik \cdot x}$. By exercise 18 in [1], the set $\{e_k : k \in \mathbb{Z}^n\}$ is orthonormal, i.e. $\langle e_k, e_n \rangle = \delta_{kn}$ (note the factor $(2\pi)^{-n}$ in the inner product). Because $L^2(Q)$ is a Hilbert space, we would like to have an orthonormal basis in it. Clearly we have one candidate, i.e. the set $\{e_k : k \in \mathbb{Z}^n\}$. Let's then recall the many characterizations of orthonormal basis in a Hilbert space.

**Theorem 1.1.** *Let $E$ be an orthonormal set in Hilbert space $H$. Then the following are equivalent:*

*(a) $E$ is maximal orthonormal set with respect to set inclusion*

*(b) if $x \in H$ such that $x \perp E$, then $x = 0$*

*(c) $\langle E \rangle$ is dense in $H$*

*(d) if $x \in H$, then $x = \sum_{e \in E} \langle x, e \rangle e$, where convergence is w.r.t. the induced norm*

*(e) $\|x\|^2 = \sum_{e \in E} |\langle x, e \rangle|^2$ for every $x \in H$*

*(f) $\langle x, y \rangle = \sum_{e \in E} \langle x, e \rangle \overline{\langle y, e \rangle}$ for every $x, y \in H$.*

*Proof.* Proof is usually done in a course of functional analysis. See for example [2] p.155 theorem 9.12. □

If any of the conditions (a)-(f) holds, we say that $E$ is an orthonormal basis/Hilbert basis for $H$, or that $E$ is a complete orthonormal set. Existence of such set $E$ is guaranteed by Zorn's lemma for every non-zero inner product space. Such sets also have the same cardinality.

Let $\ell^2(\mathbb{Z}^n)$ be the space of complex sequences $\omega : \mathbb{Z}^n \to \mathbb{C}$ with inner product

$$(\omega|\tau) = \sum_{k \in \mathbb{Z}^n} \omega(k)\overline{\tau(k)}$$

and norm $\|\omega\| = \sqrt{(\omega|\omega)}$. For the moment, let's believe that the set $\{e_k : k \in \mathbb{Z}^n\}$ is complete in $L^2(Q)$. Then we get:

(i) Condition (d) gives us representation of any function $f \in L^2(Q)$ as a Fourier-sum

$$f = \sum_{k \in \mathbb{Z}^n} \langle f, e_k \rangle e_k \quad \text{or} \quad f(x) = \sum_{k \in \mathbb{Z}^n} \widehat{f}(k) e^{ik \cdot x},$$

where $\widehat{f} : \mathbb{Z}^n \to \mathbb{C}$ is defined as

$$\widehat{f}(k) = \langle f, e_k \rangle = \frac{1}{(2\pi)^n} \int_Q f(x) e^{-ik \cdot x} \mathrm{d}x. \tag{1}$$

Remember that this convergence is in $L^2$-sense; we haven't discussed anything about pointwise convergence.

(ii) Condition (e) tells that norms are preserved between $f$ and $\widehat{f}$ since

$$\|f\|^2 = \sum_{k \in \mathbb{Z}^n} |\langle f, e_k \rangle|^2 = \sum_{k \in \mathbb{Z}^n} |\widehat{f}(k)|^2 = \|\widehat{f}\|^2.$$

Therefore every $f \in L^2(Q)$ determines a function $\widehat{f} \in \ell^2(\mathbb{Z}^n)$ through equation (1). The converse is also true: if we have $\omega \in \ell^2(\mathbb{Z}^n)$, then we can set $f : Q \to \mathbb{C}$, $f(x) = \sum_{k \in \mathbb{Z}^n} \omega(k) e^{ik \cdot x}$. This defines a function on $Q$, because the sequence

$$f_n(x) = \sum_{\substack{k \in \mathbb{Z}^n \\ |k| \leq n}} \omega(k) e^{ik \cdot x}$$

is Cauchy in $L^2(Q)$, which is a complete normed space.

The previous discussion suggests us to define the following operator:

**Definition 1.2.** The Fourier transform on $Q$ is a linear operator $\mathcal{F} : L^2(Q) \to \ell^2(\mathbb{Z}^n)$ defined as

$$\mathcal{F}(f)(k) = \widehat{f}(k) = \frac{1}{(2\pi)^n} \int_Q f(x) e^{-ik \cdot x} \mathrm{d}x.$$

We immediately get the next result:

**Theorem 1.3.** *The Fourier transform $\mathcal{F}$ defined as above is an unitary isometry. The inverse Fourier transform $\mathcal{F}^{-1} : \ell^2(\mathbb{Z}^n) \to L^2(Q)$,*

$$\mathcal{F}^{-1}(\omega)(x) = \sum_{k \in \mathbb{Z}^n} \omega(k) e^{ik \cdot x},$$

*is also an unitary isometry.*

*Proof.* $\mathcal{F}$ is clearly injective, because $\mathcal{F}(f) = 0$ implies $\langle f, e_k \rangle = 0$ for all $k \in \mathbb{Z}^n$ and hence $f = 0$ by condition (b). Define $T : \ell^2(\mathbb{Z}^n) \to L^2(Q)$, $T(\omega)(x) = \sum_{k \in \mathbb{Z}^n} \omega(k) e^{ik \cdot x}$. Then straight computation shows that $T$ is the right inverse of $\mathcal{F}$, which means that $\mathcal{F}$ is surjective and thus bijective. Hence $T = \mathcal{F}^{-1}$. Condition (e) says that $\mathcal{F}$ is an isometry, see previous discussion (ii). Therefore also $\mathcal{F}^{-1}$ is an isometry. From the definition of the adjoint operator one can see that $\langle f, \mathcal{F}^*(\omega) \rangle = (\mathcal{F}(f)|\omega) = \langle f, \mathcal{F}^{-1}(\omega) \rangle$. Thus $\mathcal{F}$ is unitary and so is $\mathcal{F}^{-1}$. $\square$

We see that all the desired properties of Fourier transform $\mathcal{F}$ will follow, if we can show that the set $\{e_k : k \in \mathbb{Z}^n\}$ forms orthonormal basis for $L^2(Q)$. To prove it we first need a couple of lemmas and definitions.

**Definition 1.4.** A sequence $K_n$ of $2\pi$-periodic continuous functions on $\mathbb{R}$ is called an approximate identity, if

(i) $K_n \geq 0$ for all $n \in \mathbb{N}$

(ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\mathrm{d}x = 1$ for all $n \in \mathbb{N}$

(iii) for every $\delta > 0$ it holds that

$$\lim_{n\to\infty} \sup_{\delta \leq |x| \leq \pi} K_n(x) = 0.$$

So for large $n$ the function $K_n$ resembles the Dirac comb. Approximate identities, as the name suggests, are used to approximate $L^p$-functions via convolution.

**Definition 1.5.** The convolution of $2\pi$-periodic functions $f$ and $g$ is defined to be

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\mathrm{d}y = -(g * f)(x),$$

which is also a $2\pi$-periodic function.

**Lemma 1.6.** *Let $K_n$ be an approximate identity. If $f \in L^p([-\pi,\pi])$ and $1 \leq p < \infty$, or if $f$ is continuous $2\pi$-periodic function and $p = \infty$, then*

$$\lim_{n\to\infty} \|K_n * f - f\|_{L^p([-\pi,\pi])} = 0,$$

*where the $L^p$-norm of a function $g \in L^p([-\pi,\pi])$ is defined as*

$$\|g\|_{L^p([-\pi,\pi])} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x)|^p dx\right)^{\frac{1}{p}}.$$

*Proof.* The proof is quite technical. Case $p = \infty$ is easier, case $1 \leq p < \infty$ needs measure-theoretic tools (Lusin's theorem and integral form of the Minkowski inequality). Details can be found in [3]. $\square$

Luckily in $\mathbb{R}$ there exists an approximate identity, which consists of trigonometric polynomials:

**Lemma 1.7.** *The sequence $Q_n : \mathbb{R} \to \mathbb{R}$,*

$$Q_n(x) = c_n\left(\frac{1+\cos x}{2}\right)^n, \quad c_n = 2\pi\left(\int_{-\pi}^{\pi}\left(\frac{1+\cos x}{2}\right)^n dx\right)^{-1},$$

*is an approximate identity on the real line.*

*Proof.* The first two properties are clear. For the third one, estimate constant $c_n$ from above to get $c_n \leq \frac{\pi(n+1)}{2}$. Then use the fact that $Q_n'(x) \leq 0$ when $x \in ]0,\pi[$ and $Q_n'(x) \geq 0$ when $x \in ]-\pi, 0[$. Thus for $\delta \leq |x| \leq \pi$ we have

$$Q_n(x) \leq Q_n(\delta) = c_n \left(\frac{1+\cos\delta}{2}\right)^n \leq \frac{\pi(n+1)}{2}\left(\frac{1+\cos\delta}{2}\right)^n \to 0$$

since $\frac{1+\cos\delta}{2} < 1$ when $0 < \delta \leq \pi$. □

**Remark 1.8.** Lemmas 1.6 and 1.7 give the classical result due to Weierstrass: trigonometric polynomials are dense in the space of continuous $2\pi$-periodic functions on the interval $[-\pi,\pi]$ equipped with the sup-norm. This can be seen by choosing the polynomial to be $P = Q_n * f$.

Now we have the right tools to prove the next theorem, which has an essential role in the proof of the main result 1.3.

**Theorem 1.9.** *If $f \in L^2(Q)$ and $\langle f, e_k\rangle = 0$ for all $k \in \mathbb{Z}^n$, then $f = 0$. Hence by condition (b) the set $\{e_k : k \in \mathbb{Z}^n\}$ forms orthonormal basis for $L^2(Q)$.*

*Proof.* Assume first $n = 1$. Let $f \in L^2([-\pi,\pi])$ be such that $\langle f, e_k\rangle = 0$ for all $k \in \mathbb{Z}$. Then the inner product between $f$ and any trigonometric polynomial $\sum_{k=-m}^{m} c_k e_k$ vanishes. Especially for any $x \in [-\pi,\pi]$ we have

$$0 = \langle f, Q_n(x-\cdot)\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\underbrace{\overline{Q_n(x-y)}}_{\in\mathbb{R}}\,\mathrm{d}y = (f * Q_n)(x) = -(Q_n * f)(x).$$

Lemmas 1.6 and 1.7 give us that

$$0 \leftarrow \|Q_n * f - f\|_{L^2([-\pi,\pi])} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\underbrace{(Q_n * f)(x)}_{=0} - f(x)|^2\right)^{\frac{1}{2}} = \|f\|.$$

Therefore $f = 0$ and the set $\{e_k : k \in \mathbb{Z}\}$ is complete in $L^2([-\pi,\pi])$. Let us then assume that the claim holds for some $n = m-1$. Define a function

$$g(x_1) = \int_{[-\pi,\pi]^{m-1}} f(x) e^{-i(k_2x_2+\ldots+k_mx_m)}\mathrm{d}x_2\ldots\mathrm{d}x_m.$$

By Cauchy-Schwartz inequality one can see that $g \in L^2([-\pi,\pi])$. Now the condition $\langle f, e_k\rangle = 0$ for all $k \in \mathbb{Z}^m$ in terms of $g$ means that

$$\begin{aligned} 0 &= \int_{[-\pi,\pi]^m} f(x)e^{-i(k_1x_1+\ldots+k_mx_m)}\mathrm{d}x_1\ldots\mathrm{d}x_m \\ &\overset{Fubini}{=} \int_{-\pi}^{\pi}\int_{[-\pi,\pi]^{m-1}} f(x)e^{-i(k_2x_2+\ldots+k_mx_m)}\mathrm{d}x_2\ldots\mathrm{d}x_m\, e^{-ik_1x_1}\mathrm{d}x_1 \\ &= \int_{-\pi}^{\pi} g(x_1)e^{-ik_1x_1}\mathrm{d}x_1 = \langle g, e_{k_1}\rangle. \end{aligned}$$

Thus $\langle g, e_{k_1} \rangle = 0$ for all $k_1 \in \mathbb{Z}$ and by the case $n = 1$ it must be that $g = 0$. This implies that for all $x_1 \in [-\pi, \pi]$

$$\int_{[-\pi,\pi]^{m-1}} f(x) e^{-i(k_2 x_2 + \ldots + k_m x_m)} \mathrm{d}x_2 \ldots \mathrm{d}x_m = 0.$$

The induction assumption then gives that $f(x_1, \ldots, x_n) = 0$ for fixed $x_1$. But since this holds for all $x_1 \in [-\pi, \pi]$, it has to be that $f = 0$, which proves the theorem. $\square$

## References


[1] Joonas Ilmavirta, Analysis and X-ray tomography, lecture notes, users.jyu.fi/ jojapeil/opetus/mats4300xrt/mats4300-v3.pdf.

[2] George Bachman and Lawrence Narici, Functional Analysis, Dover publications 2000.

[3] Mikko Salo, Fourier analysis and distribution theory, lecture notes, users.jyu.fi/salomi/lecturenotes/FA_distributions.pdf.