Fourier transform conventions

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Abstract

We present different conventions for the Fourier transform in $\mathbb{R}^n$, their properties, and their relations.

1 Assumptions

All parameters are positive real. α is a multi index. The variable in the direct space is usually x, and in the momentum space usually k.

2 Definitions

Define the Fourier transform of $f : \mathbb{R}^n \to \mathbb{C}$ by

$$\mathcal{F}f(k) = A \int_{\mathbb{R}^n} e^{-iBk\cdot x} f(x)\mathrm{d}x \tag{1}$$

and the inverse Fourier transform by

$$\mathcal{F}^{-1}f(x) = A^* \int_{\mathbb{R}^n} e^{iB^*k\cdot x} f(k)\mathrm{d}k. \tag{2}$$

Define convolution by

$$(f * g)(x) = C \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}y \tag{3}$$

on direct side and similarly with C^* on the momentum side. The inner product is

$$\langle f|g\rangle = D \int_{\mathbb{R}^n} f(x)\bar{g}(x)\mathrm{d}x \tag{4}$$

on direct side and similarly with D^* on the momentum side. The convolution and inner product satisfy

$$\begin{aligned} \mathcal{F}(f * g) &= E\mathcal{F}f \cdot \mathcal{F}g, \\ \mathcal{F}(f \cdot g) &= F\mathcal{F}f * \mathcal{F}g, \\ \langle \mathcal{F}f|\mathcal{F}g\rangle &= G\langle f|g\rangle . \end{aligned} \tag{5}$$

3 Derived properties

Define operators a, a^α (multiplication by argument), ∂^α (derivative), t_y (translation) amd s_μ (scaling) by

$$\begin{aligned}
\varphi(a)f(x) &= \varphi(x)f(x) \quad \text{for a function } \varphi, \\
a^\alpha f(x) &= x^\alpha f(x), \\
\partial^\alpha f(x) &= \frac{\partial^{|\alpha|}}{\partial x^\alpha} f(x), \\
t_y f(x) &= f(x-y), \\
s_\mu f(x) &= f(\mu x).
\end{aligned} \tag{6}$$

We have

$$\begin{aligned}
\mathcal{F}^{\pm 1} \circ s_\mu &= |\mu|^{-1} s_{\mu^{-1}} \circ \mathcal{F}^{\pm 1}, \\
\mathcal{F} \circ t_y &= e^{-iBy\cdot a} \circ \mathcal{F}, \\
\mathcal{F}^{-1} \circ t_y &= e^{iB^* y\cdot a} \circ \mathcal{F}^{-1}, \\
\partial^\alpha \circ \mathcal{F} &= (-iB)^{|\alpha|} \mathcal{F} \circ a^\alpha, \\
\partial^\alpha \circ \mathcal{F}^{-1} &= (iB^*)^{|\alpha|} \mathcal{F}^{-1} \circ a^\alpha, \\
\mathcal{F} \circ \partial^\alpha &= (iB)^{|\alpha|} a^\alpha \circ \mathcal{F}, \\
\mathcal{F}^{-1} \circ \partial^\alpha &= (-iB^*)^{|\alpha|} a^\alpha \circ \mathcal{F}^{-1}.
\end{aligned} \tag{7}$$

Also, for constant and delta functions

$$\begin{aligned}
\mathcal{F}1(k) &= \frac{1}{A^*}\delta(k), \\
\mathcal{F}\delta(k) &= A.
\end{aligned} \tag{8}$$

4 Relations between parameters

To have $\mathcal{F} \circ \mathcal{F}^{-1} = \mathrm{id}$, we need to have

$$\begin{aligned}
B &= B^*, \\
B^n &= (2\pi)^n A A^*.
\end{aligned} \tag{9}$$

The convolution results are

$$\begin{aligned}
E &= C/A, \\
F &= A^*/C^*.
\end{aligned} \tag{10}$$

For Parseval's theorem, we get

$$G = \frac{AD^*}{A^* D}. \tag{11}$$

There are in total 6 free parameters: A, B, C, C^*, D, D^*.

5 Relations between conventions

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two Fourier transforms with similarly indexed parameters. Then

$$
\begin{aligned}
\mathcal{F}_1 &= \frac{A_1}{A_2}\left(\frac{B_2}{B_1}\right)^n \mathcal{F}_2 \circ s_{B_2/B_1} = \frac{A_1}{A_2}\left(\frac{B_2}{B_1}\right)^{n-1} s_{B_1/B_2} \circ \mathcal{F}_2, \\
\mathcal{F}_1^{-1} &= \frac{A_1^*}{A_2^*}\left(\frac{B_2^*}{B_1^*}\right)^n \mathcal{F}_2^{-1} \circ s_{B_2^*/B_1^*} = \frac{A_1^*}{A_2^*}\left(\frac{B_2^*}{B_1^*}\right)^{n-1} s_{B_1^*/B_2^*} \circ \mathcal{F}_2^{-1}, \\
*_1 &= \frac{C_1}{C_2} *_2 \quad \text{direct}, \\
*_1 &= \frac{C_1^*}{C_2^*} *_2 \quad \text{momentum}, \\
\langle\cdot|\cdot\rangle_1 &= \frac{D_1}{D_2} \langle\cdot|\cdot\rangle_2 \quad \text{direct}, \\
\langle\cdot|\cdot\rangle_1 &= \frac{D_1^*}{D_2^*} \langle\cdot|\cdot\rangle_2 \quad \text{momentum}.
\end{aligned}
\tag{12}
$$

6 Examples

In the table below, $\tau_h = (2\pi h)^{n/2}$ and $\tau = \tau_1$. Usually h stands for Planck's constant $\hbar$, but it may also be the semiclassical parameter.

	A	A^*	B	B^*	C	C^*	D	D^*	E	F	G
QM	τ_h^{-1}	τ_h^{-1}	h^{-1}	h^{-1}	1	1	1	1	τ_h	τ_h^{-1}	1
scaled QM	τ^{-1}	τ^{-1}	1	1	1	1	1	1	τ	τ^{-1}	1
QFT	1	τ^{-2}	1	1	1	τ^{-2}	1	τ^{-2}	1	1	1
symmetric	τ^{-1}	τ^{-1}	1	1	τ^{-1}	τ^{-1}	1	1	1	1	1
frequency	1	1	2π	2π	1	1	1	1	1	1	1
inverted	A^*	A	$-B^*$	$-B$	C^*	C	D^*	D	F	E	G^{-1}
general	A	$\frac{B^n}{A\tau^2}$	B	B	C	C^*	D	D^*	C/A	$\frac{B^n}{AC^*\tau^2}$	$\frac{D^*A^2\tau^2}{DB^n}$