A Posteriori Methods

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Table of Contents

1. General Properties of a Posteriori Methods
2. Quality Measures for Representations of the PO set
3. Examples of a Posteriori Methods
4. Developing more a Posteriori Methods
5. Critical Evaluation of a Posteriori Methods
6. From a Posteriori Methods to Interactive Methods
A Posteriori Methods

- In a posteriori methods,
  1. a representative set of/all the Pareto optimal (PO) solutions is generated and
  2. the decision maker chooses the best one

- Most a posteriori methods fall into two classes:
  "Classical methods" solve multiple single-objective optimization problems that each produce a PO solution
  Evolutionary methods mimic natural evolution and try to evolve a "population of solutions" simultaneously into a representative set of PO solutions

- Today’s lecture deals with "classical methods"
Unlike a priori methods, most classical a posteriori methods are problem dependent
- Some work only with linear problems, some with convex and others make further assumptions

In addition, a posteriori methods aim either at producing the complete set of PO solutions or a representation/approximation of it
- A representation of the Pareto front/Pareto optimal set is a finite set of PO solutions that is "dense enough"
- An approximation may contain vectors that are not PO solutions but merely approximate them
- The definitions above may, however, vary
Quality Measures for Representations of the PO Set

- When one produces representations of the PO set, one needs quality measures.
- As argued by Wiecek and Faulkenberg in 2010, these measures (for classical algorithms) measure one the following:
  1. cardinality (i.e., number of PO solutions produced),
  2. coverage (i.e., covering the complete PO set), or
  3. spacing (i.e., how equally spaced the solutions are).
- The above assumes that each solution in the representation is actually PO.
- For classical algorithms, this can be assumed since, if the resulting single objective optimization problems are solved correctly, then the resulting solutions are PO.
- However, for evolutionary algorithms this cannot be assumed and, thus, for them must also measure the distance to the actual PO set.
An Example of Quality Measures: Hypervolume

- Introduced by Zitzler and Thiele in 1998
- Measures simply the multidimensional volume (in the objective space) of the hypercubes that include solutions that
  1. are dominated by a solution in the representation and
  2. dominate the nadir solution estimated from the current representation

**Figure from** [http://ls11-www.cs.uni-dortmund.de/detail/rudolph/hypervolume/hv.png?id=rudolph%3Ahypervolume%3Astart](http://ls11-www.cs.uni-dortmund.de/detail/rudolph/hypervolume/hv.png?id=rudolph%3Ahypervolume%3Astart)
Developed by Haimes et al. in 1971
Choose a small $\delta > 0$ and $j \in \{1, 2\}$
Let $\text{opt}(\epsilon)$ be the optimal solution of the (augmented) $\epsilon$-constraint problem with constraints given by $\epsilon \in \mathbb{R}^2$
Then the procedure is as follows

Set $P := \emptyset$ and $\epsilon := z^{\text{nadir}}$;
while $\epsilon > z^{\text{ideal}}$ do
  Set $x := \text{opt}(\epsilon - \delta)$;
  Set $P := P \cup \{x\}$;
  Set $\epsilon_i := \epsilon_i - \delta$ for all $i = 1, 2$;
end

Has been generalized to problems with more objectives by Laumans, Thiele and Zitzler in 2005
In addition, there is a so-called Improved $\epsilon$-constraint method by Ehrgoogt and Ruzika in 2008
An illustration of the $\epsilon$-Constraint Scheme with Two-Objective Problems
Properties of the $\epsilon$-Constraint Scheme in Two-Objective Problems

+ Guarantees maximal value for "gaps" in the constrained objective, but
  - the distance of solutions in the minimized objective may be big (see figure in the previous slide)

+ When the minimal distance between two PO solutions is greater than zero, then the method can be modified to find all the PO solutions.
Normal Boundary Intersection

- Developed by Das and Dennis in 1998
- Is based on constructing the convex hull of vectors in the objective space which minimize the objective functions individually (so-called convex hull of individual minima, CHIM) and selecting an equally spaced set on that
- For each vector in the equally spaced set, we find the intersection of the normal of the CHIM originating from the vector and the boundary of the feasible objective set \( f(S) \)
- For a vector \( \nu \) in the CHIM, the intersection is found by solving the optimization problem

\[
\begin{align*}
\min & \quad t \\
\text{s.t.} & \quad f(x) = \nu + t \mathbf{1} \\
& \quad x \in S,
\end{align*}
\]

where \( \mathbf{1} = (1, \ldots, 1)^T \).
Illustration of the Normal Boundary Intersection
In problems with 2-objectives and connected Pareto optimal sets, the Normal Boundary Intersection method produces rather evenly spaced representations.

- In problems with more than 3 objectives, the method may miss parts of the Pareto front.
- Does not always produce Pareto optimal solutions, nor is the feasible set always non-empty.
- The CHIM is hard to construct, if two individual minima are the same.
ADBASE by Steuer

- Developed by Steuer in 1975
- Uses the fact that in linear multiobjective optimization problems
  1. the feasible objective set is a convex polyhedral set
  2. the Pareto front is a union of nondominated facets in the polyhedral set
- Is based on pivoting from one Pareto optimal solution to another adjacent one (similarly to the single-objective Simplex method)
Properties of the ADBASE

- Only works with linear problems
+ Can find all the nondominated extreme vertices
- Does not find the nondominated facets (which may be hard in higher dimensions)
+ Has still (!) a working implementation that has been actively developed
Other a Posteriori Methods

- Normal constraint method by Messac and Mattson in 2004
- Benson’s algorithm for linear multiobjective problems in 1998
- Feasible and Reasonable goals methods of Lotov et al for producing Pareto front approximations in 2006
- Directed search domain by Erfani and Utyuzhnikov in 2011
On Developing new a Posteriori Methods

- Basically, any a priori method can be turned into an a posteriori method by producing an evenly spaced set in the set of preferences.
- However, the above approach rarely produces good quality representations.
- In more advanced a posteriori methods, there is a "backward" loop between solving the optimization problems that produce the PO solutions and formulating the optimization problems.
- For example, in the $\epsilon$-constraint scheme, the next $\epsilon_i$ can be set to $f_i(x) - \delta$ for each found PO solution $x$.
- The loop allows to produce better quality representations and solve fewer optimization problems or sometimes even find the complete PO set.
Common Problems with A posteriori Methods

- It may be hard for the DM to choose from a large representation, or approximation or the complete Pareto optimal set.
- Computing the representation, or approximation or the complete Pareto optimal set may take time.
- It may be hard to know beforehand how big a representation is dense enough.
  - This actually requires knowledge about the preferences of the DM.
- Visualization of the complete set of Pareto optimal solutions is easy only in problems with 2 or 3 objectives.
In interactive methods, the optimization and DM’s preference articulation take turns

**Pluses**
- As the DM may guide the search, only PO solutions that are interesting to him are computed
- The burden to the DM gets lighter, as the DM only evaluates few solutions at the time

**Minuses**
- Need active participation from the DM (who may be busy)
- New demands for the computational time, because the DM is waiting while computation is done
- In method development, one must take into account that the DM’s preferences may evolve because of e.g., learning
- A need to develop graphical user interfaces