Exercise 8 Return by 14.00 on Tue Nov 12, discussed Wed Nov 13 at 12:15 in FYS5

1. As on p. 247-8 in the lecture notes, let's study the group $SU(N)$.

   (a) Knowing that the $SU(N)$ group matrices can be written in the form

   $$U(\alpha) = \exp\left[i\sum_{i=1}^{d}\alpha_i T_i\right],$$

   where T_i are the group generator matrices and α_i are real parameters, convince yourself that the generators have to be hermitian and traceless. You can use the facts $\det(e^A) = e^{\mathrm{Tr}(A)}$ and $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$. The purpose of this problems is that you understand why the color symmetry group $SU(3)$ generators (The Gell-Mann -matrices) satisfy these conditions.

   (b) The generators are linearly independent matrices and fulfil $\mathrm{Tr}(T_iT_j) = \lambda\delta_{ij}$. Using the Lie algebra for this group and the properties of the trace, show that the structure constants can be expressed as $C_{ljk} = -\frac{i}{\lambda}\mathrm{Tr}(T_l[T_j,T_k])$ and that C_{ljk} are fully antisymmetric in any mutual exchange of indices.

2. As suggested on p.249 in the lecture notes,

   (a) Generalize the Noether theorem for the globally $U(1)$ symmetric Lagrangian

   $$\mathcal{L}(\phi,\phi^*,\partial_\mu\phi,\partial_\mu\phi^*) = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - V_I(\phi^*\phi),$$

   describing a charged scalar field (see p. 249), i.e. find the form of the conserved 4-current, analogous to that on p. 236.

   (b) Using the $U(1)$ transformation $U = e^{i\theta}$ (expand), find the variations $\delta\phi$ and $\delta\phi^*$ and based on your result above, show that you arrive at the same conserved 4-current as given on p. 249.

3. Using the general properties of the Dirac matrices α^1, α^2, α^3 and β (see p. 255 in the lecture notes), and the definitions of the Dirac gamma matrices γ^μ (p. 256), verify the following, representation-independent, results:

   (a) $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_4$, known as the Clifford algebra,

   (b) $\gamma^i = -\gamma^0\gamma^i\gamma^0$,

   (c) $\gamma^{0\dagger} = \gamma^0$ ja $\gamma^{i\dagger} = \gamma^0\gamma^i\gamma^0 = -\gamma^i$,

   (d) $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.

4. Using the explicit 2-block forms for the free spin-$\frac{1}{2}$ particle & antiparticle spinors $u^{(r)}(p)$ and $v^{(r)}(p)$ $(r, s = 1, 2)$, which we derived in the lecture notes (p.259-263), show that with the chosen normalization $\int_V \mathrm{d}^3x\ \rho = 2E_p$ (p. 264), we have the following

   (a) normalization constant: $N(\mathbf{p}) = \sqrt{E_p + m}$
   (Hints: Do the spinor and matrix multiplications using the 2-block forms. Recall the Pauli spin-matrix property $(\vec{\sigma} \cdot \vec{p})^2 = \vec{p}^2$)

   (b) orthogonality relations: $u^{(r)\dagger}(p)u^{(s)}(p) = v^{(r)\dagger}(p)v^{(s)}(p) = 2E_p\delta_{rs}$

   (c) projection operators:
   (Hint: do the multiplication by γ^0 only after performing the spin sums)

$$\sum_{s=1,2} u^{(s)}(p)\overline{u}^{(s)}(p) = \not{p} + m$$

$$\sum_{s=1,2} v^{(s)}(p)\overline{v}^{(s)}(p) = \not{p} - m$$

5. Let's practice drawing Feynman graphs for scatterings in a given theory

   (a) Let's first consider a ϕ^3-theory for an interacting neutral spin-0 particle, with a Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3.$$

   Draw all possible topologically different Feynman diagrams for $2 \to 2$ scattering in this theory, first in the lowest order (LO) in λ, then in the next-to-lowest order (NLO). Note that in the latter case there are quite many diagrams, be systematic in sorting these out.

   (b) Let's then consider some QED scatterings. Draw all Feynman graphs which contribute to the invariant amplitude in the lowest order in e (figure out which power of e corresponds to the lowest order in each case!) for the following scatterings

      i $e^+ + e^- \to e^+ + e^-$

      ii $e^+ + e^- \to \mu^+ + \mu^-$

      iii $e^+ + e^- \to e^+ + e^- + \mu^+ + \mu^-$

   Label the lines and draw the required arrows.