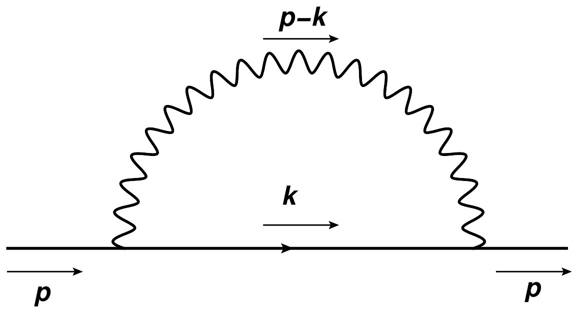
This still contains part of the IR divergence and the UV infinity from the vertex loop. In the next section we sort out their destiny.

7.3 The electron self energy

Based on the LSZ theorem the external electron legs are to be multiplied by the renormalization constant $\sqrt{Z}$ defined as the pole of the full propagator. The first QED contribution is given by the following diagram:



As part of a larger diagram, this piece will correspond to an expression,

$$\int \frac{d^4k}{(2\pi)^4}\frac{i(\not{p}+m)}{p^2-m^2}(-ie\gamma_\mu)\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}(-ie\gamma_\nu)\frac{i(\not{p}+m)}{p^2-m^2}\frac{-ig^{\mu\nu}}{(p-k)^2+i\epsilon}$$

$$=\frac{i(\not{p}+m)}{p^2-m^2}\left[-e^2\int\frac{d^4k}{(2\pi)^4}\gamma_\mu\frac{(\not{k}+m)}{k^2-m^2+i\epsilon}\gamma^\mu\frac{1}{(p-k)^2+i\epsilon}\right]\frac{i(\not{p}+m)}{p^2-m^2}$$

$$=\frac{i(\not{p}+m)}{p^2-m^2}\left[-i\Sigma_2(p)\right]\frac{i(\not{p}+m)}{p^2-m^2}\,, \tag{7.59}$$

when we define

$$-i\Sigma_2(p)\equiv -e^2\int\frac{d^4k}{(2\pi)^4}\gamma_\mu\frac{\not{k}+m}{k^2-m^2+i\epsilon}\gamma^\mu\frac{1}{(p-k)^2+i\epsilon}\,. \tag{7.60}$$

We can process the loop integral with the methods of the previous section. By using the Feynman parameters,

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(p-k)^2+i\epsilon} \tag{7.61}$$

$$=\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\frac{1}{\left[k^2-2x(p\cdot k)+xp^2-(1-x)m^2+i\epsilon\right]^2}\,.$$

Completing the square, $k^2 - 2x(p\cdot k) = (k-xp)^2 - x^2p^2$, and defining a new integration variable $\ell \equiv k - xp$,

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(p-k)^2+i\epsilon} = \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^2},$$

where $\Delta = -x(1-x)p^2 + (1-x)m^2$. In the numerator,

$$\gamma_\mu(\not{k}+m)\gamma^\mu = -2\not{k} + 4m \to -2(\not{\ell}+x\not{p}) + 4m\,. \tag{7.62}$$

Dropping the term linear in ℓ (integrates to zero), we get

$$\int \frac{d^4k}{(2\pi)^4}\frac{\gamma_\mu(\not{k}+m)\gamma^\mu}{k^2-m^2+i\epsilon}\frac{1}{(p-k)^2+i\epsilon} = \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{-2x\not{p}+4m}{\left[\ell^2-\Delta+i\epsilon\right]^2}\,.$$

When k is large, the integral behaves as $\int d^4k/k^4$ which gives a logarithmic UV divergence. In addition, in the limit $x \to 1$ we see that $\Delta \to 0$ which yields an IR divergence. We regulate these using the same technique as in the case of vertex correction. The infrared divergence gets regulated by giving the photon a small mass μ^2,

$$\frac{1}{(p-k)^2+i\epsilon} \to \frac{1}{(p-k)^2-\mu^2+i\epsilon}\,, \tag{7.63}$$

and the Pauli-Villars prescription removes the UV divergence when we include the subtraction term,

$$\frac{1}{(p-k)^2+i\epsilon} \to \frac{1}{(p-k)^2-\mu^2+i\epsilon} - \frac{1}{(p-k)^2-\Lambda^2+i\epsilon}\,. \tag{7.64}$$

Doing this,

$$\int \frac{d^4k}{(2\pi)^4}\frac{\gamma_\mu(\not{k}+m)\gamma^\mu}{k^2-m^2+i\epsilon}\frac{1}{(p-k)^2+i\epsilon} \longrightarrow \tag{7.65}$$

$$\int_0^1 dx\,\left(-2x\not{p}+4m\right)\int\frac{d^4\ell}{(2\pi)^4}\left[\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^2} - \frac{1}{\left[\ell^2-\Delta_\Lambda+i\epsilon\right]^2}\right],$$

where now

$$\Delta = -x(1-x)p^2 + (1-x)m^2 + x\mu^2\,, \tag{7.66}$$

$$\Delta_\Lambda = -x(1-x)p^2 + (1-x)m^2 + x\Lambda^2\,. \tag{7.67}$$

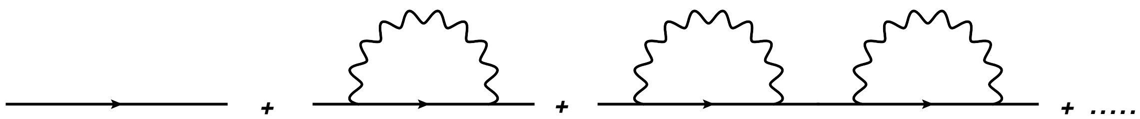
By doing the Wick rotation,

$$\int \frac{d^4\ell}{(2\pi)^4}\left[\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^2}-\frac{1}{\left[\ell^2-\Delta_\Lambda+i\epsilon\right]^2}\right]=\frac{i}{(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right),$$

so overall, when $\Lambda^2 \to \infty$, we find

$$\boxed{\Sigma_2(p)=\frac{\alpha}{2\pi}\int_0^1 dx\,(-x\not{p}+2m)\log\left(\frac{x\Lambda^2}{-x(1-x)p^2+(1-x)m^2+x\mu^2}\right)} \tag{7.68}$$

We now proceed as in Section 5, and sum the contribution of the just-computed diagram to all orders. At this point we should also remember that the mass m in the above expression should actually be the unphysical bare mass m_0.



This diagrammatic sum corresponds to an expression,

$$\begin{aligned}&\frac{i(\not{p}+m_0)}{p^2-m_0^2}+\frac{i(\not{p}+m_0)}{p^2-m_0^2}\left[-i\Sigma_2(p)\right]\frac{i(\not{p}+m_0)}{p^2-m_0^2} \\ &\quad+\frac{i(\not{p}+m_0)}{p^2-m_0^2}\left[-i\Sigma_2(p)\right]\frac{i(\not{p}+m_0)}{p^2-m_0^2}\left[-i\Sigma_2(p)\right]\frac{i(\not{p}+m_0)}{p^2-m_0^2}+\cdots\end{aligned} \tag{7.69}$$

By using a shorter form,

$$\frac{i(\not{p}+m_0)}{p^2-m_0^2}=\frac{i}{\not{p}-m_0}, \tag{7.70}$$

where $(\not{p}-m_0)^{-1}$ refers to the inverse of $(\not{p}-m_0)$ we can write the above sum as,

$$\frac{i}{\not p - m_0} + \frac{i}{\not p - m_0}\left[-i\Sigma_2(p)\right]\frac{i}{\not p - m_0} \tag{7.71}$$

$$+\frac{i}{\not p - m_0}\left[-i\Sigma_2(p)\right]\frac{i}{\not p - m_0}\left[-i\Sigma_2(p)\right]\frac{i}{\not p - m_0} + \cdots$$

$$=\frac{i}{\not p - m_0} + \frac{i}{\not p - m_0}\frac{\Sigma_2(p)}{\not p - m_0} + \frac{i}{\not p - m_0}\left(\frac{\Sigma_2(p)}{\not p - m_0}\right)^2 + \cdots$$

$$=\frac{i}{\not p - m_0}\left[1 + \frac{\Sigma_2(p)}{\not p - m_0} + \left(\frac{\Sigma_2(p)}{\not p - m_0}\right)^2 + \cdots\right].$$

where we used the fact that $(\not p - m_0)$ and its inverse commute with $\Sigma_2(p)$. Formally this is a geometric series which we can sum:

$$\frac{i}{\not p - m_0}\left[1 + \frac{\Sigma_2(p)}{\not p - m_0} + \left(\frac{\Sigma_2(p)}{\not p - m_0}\right)^2 + \cdots\right]$$

$$= \frac{i}{\not p - m_0}\frac{1}{1 - \frac{\Sigma_2(p)}{\not p - m_0}} = \frac{i}{\not p - m_0 - \Sigma_2(p)}. \tag{7.72}$$

More explicitly,

$$\frac{1}{\not p - m_0 - \Sigma_2(p)} = \frac{\not p\left[1 - \Sigma'(p^2)\right] + m_0\left[1 + \Sigma''(p^2)\right]}{p^2\left[1 - \Sigma'(p^2)\right]^2 - m_0^2\left[1 + \Sigma''(p^2)\right]^2}, \tag{7.73}$$

where

$$\Sigma'(p^2) \equiv -\frac{\alpha}{2\pi}\int_0^1 dx x \log\left(\frac{x\Lambda^2}{-x(1-x)p^2 + (1-x)m^2 + x\mu^2}\right),$$

$$\Sigma''(p^2) \equiv 2\frac{\alpha}{2\pi}\int_0^1 dx \log\left(\frac{x\Lambda^2}{-x(1-x)p^2 + (1-x)m^2 + x\mu^2}\right).$$

Based on the general discussion of Section 5 the summed propagator (7.73) should have a pole at the physical mass, $p^2 = m^2$. We find this as a solution of the equation

$$\left[p^2\left[1 - \Sigma'(p^2)\right]^2 - m_0^2\left[1 + \Sigma''(p^2)\right]^2\right]\Big|_{p^2=m^2} = 0. \tag{7.74}$$

Near this pole, the summed propagator is of the form,

$$Z_2 \frac{i(\not{p}+m)}{p^2-m^2}\,, \tag{7.75}$$

where Z_2 is the renormalization factor related to the electron field (the one that appears in the LSZ theorem). After a bit of tinkering, we find (Ex.),

$$\begin{aligned}
m^2 &= m_0^2 \times \left[\!\!\left[1 + \frac{\alpha}{2\pi}\int_0^1 dx\,(4-2x)\log\left[\frac{x\Lambda^2}{(1-x)^2 m_0^2 + x\mu^2}\right] \right]\!\!\right] \\
Z_2 &= 1 + \frac{\alpha}{2\pi}\int_0^1 dx \left[\!\!\left[-x\log\left[\frac{x\Lambda^2}{(1-x)^2m^2 + x\mu^2}\right] \right.\right. \\
&\qquad\qquad \left.\left. + (2-x)\,\frac{2m^2x(1-x)}{(1-x)^2m^2+x\mu^2} \right]\!\!\right]
\end{aligned} \tag{7.76}$$

To this order in coupling constant α, we can equally well use m^2 or m_0^2 in what is inside the double square brackets.

The results above indicate that the mass parameter m_0 that appears in the Lagrangian and what we call a physical mass m are different by a divergent factor. The physical mass m is of course finite which indicates that m_0 has to be divergent as well. The equation above implies that we should define,

$$m_0^2 = m^2 \times \left[\!\!\left[1 - \frac{\alpha}{2\pi}\int_0^1 dx\,(4-2x)\log\left[\frac{x\Lambda^2}{(1-x)^2m^2+x\mu^2}\right] \right]\!\!\right],$$

and use this definition in calculations – we recall that the quantity that appears in the Feynman rules is m_0. When doing so, part of the UV infinities coming from the loop diagrams cancel. It is important that the above definition is made only once – the same definition removes UV infinities from all kinds of processes, not just the one we have considered here. From the view point of the process considered now this does not really show up as the leading-order diagram does not contain an electron propagator. The procedure outlined here is called the **mass renormalization**. The definition of m_0 is, however, not unique and from the viewpoint of removing divergences we could add or subtract whatever finite terms. Different choices are called **schemes**. The definition above is a common one and known as the **on-shell scheme** or **pole-mass scheme**. It is also worth pointing out that in the massless case the mass renormalization is not need but $m_0 = m = 0$.

The renormalization constant Z_2 contains both UV- and IR divergences. They are easily isolated from the complete expression (7.76),

UV part:

$$\begin{aligned}
&\frac{\alpha}{2\pi}\int_0^1 dx \left[\!\!\left[-x\log\left[\frac{x\Lambda^2}{(1-x)^2m^2+x\mu^2}\right]\right]\!\!\right] \\
=&\frac{\alpha}{2\pi}\int_0^1 dx \left[\!\!\left[-x\log\left(\frac{\Lambda^2}{m^2}\right)\right]\!\!\right] + \cdots \\
=&-\frac{1}{2}\frac{\alpha}{2\pi}\log\left(\frac{\Lambda^2}{m^2}\right) + \cdots
\end{aligned} \tag{7.77}$$

IR part:

$$\begin{aligned}
&\frac{\alpha}{2\pi}\int_0^1 dx \left[\!\!\left[(2-x)\,\frac{2m^2x(1-x)}{(1-x)^2m^2+x\mu^2}\right]\!\!\right] \\
=&2m^2\frac{\alpha}{2\pi}\int_0^1 dx \left[\!\!\left[[1+(1-x)]\,\frac{[-(1-x)+1]\,(1-x)}{(1-x)^2m^2+x\mu^2}\right]\!\!\right] \\
=&2m^2\frac{\alpha}{2\pi}\int_0^1 dx \frac{(1-x)}{(1-x)^2m^2+\mu^2}\Big[1+\mathcal{O}(1-x)\Big] \\
=&\frac{\alpha}{2\pi}\log\left(\frac{m^2}{\mu^2}\right)
\end{aligned} \tag{7.78}$$

The renormalization factor Z_2 is thus,

$$Z_2 = 1 + \frac{\alpha}{2\pi}\left[-\frac{1}{2}\log\left(\frac{\Lambda^2}{m^2}\right) + \log\left(\frac{m^2}{\mu^2}\right)\right] + \text{finite terms}$$

According to the LSZ theorem each external electron enters the scattering amplitude with a factor of $\sqrt{Z_2}$, so in total we have a factor Z_2^2 multiplying the cross section. To order α,

$$Z_2^2 = 1 + \frac{\alpha}{2\pi}\left[-\log\left(\frac{\Lambda^2}{m^2}\right) + 2\log\left(\frac{m^2}{\mu^2}\right)\right] + \text{finite terms}\,.$$

The contribution of the external-leg corrections to the cross section is thus,

$$\begin{aligned} d\sigma^{\text{external leg}}(p,p') &= d\sigma^0(p,p') \times \left(Z_2^2 - 1\right) \\ &= d\sigma^0(p,p') \times \frac{\alpha}{2\pi}\left[-\log\left(\frac{\Lambda^2}{m^2}\right) + 2\log\left(\frac{m^2}{\mu^2}\right)\right] \\ &\quad + \text{finite terms} \end{aligned} \tag{7.79}$$

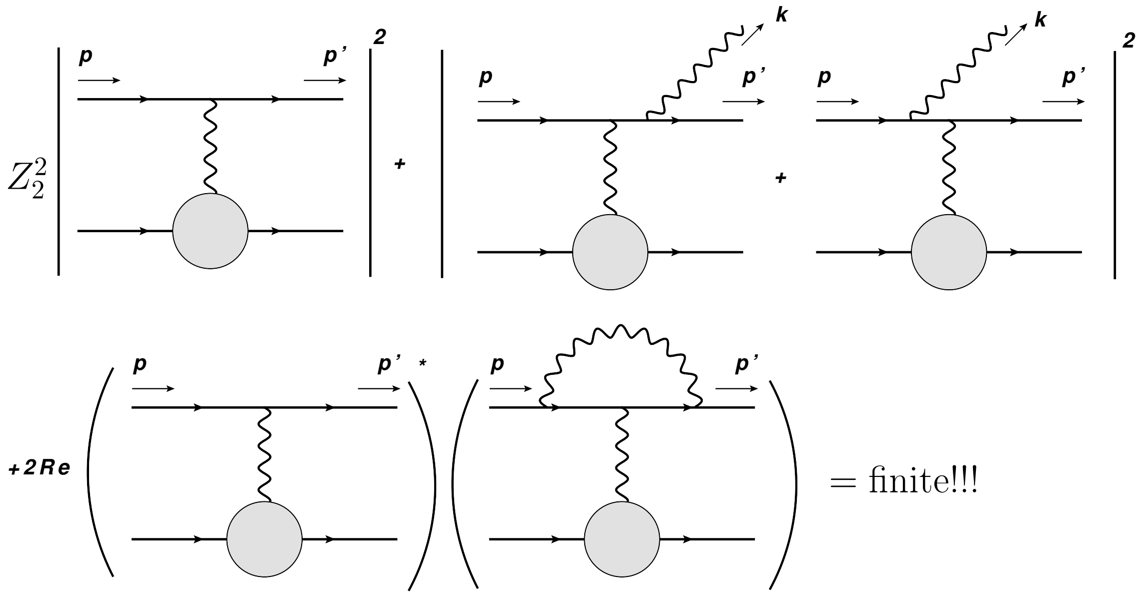
To close this section, we compare the obtained result with Eq. (7.58), the sum of braking radiation and vertex correction,

$$\begin{aligned} d\sigma^{\text{rad}}(p,p') + d\sigma^{\text{vertex}}(p,p') &= \text{finite terms} \\ + d\sigma^0(p,p') \times \frac{\alpha}{2\pi}\Bigg\{\log\left(\frac{\Lambda^2}{-q^2}\right) &- 2\log\left(\frac{-q^2}{\mu^2}\right)\Bigg\}\,. \end{aligned} \tag{7.80}$$

Remarkably, the divergence structure is exactly the same but the signs are the opposite! Thus the sum of all three contributions is finite

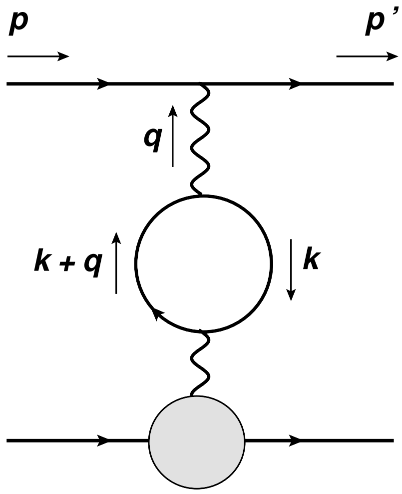
$$\begin{aligned} & d\sigma^{\text{rad}}(p,p') + d\sigma^{\text{vertex}}(p,p') + d\sigma^{\text{external leg}}(p,p') \\ &= d\sigma^0(p,p') \times \frac{\alpha}{2\pi}\left\{\log\left(\frac{m^2}{-q^2}\right) - 2\log\left(\frac{-q^2}{m^2}\right)\right\} + \cdots \\ &= \text{a finite number}\,. \end{aligned} \tag{7.81}$$

We have now seen how different radiation/loop diagrams can yield infinities but when appropriately combined, it is possible to find a finite result. The cancellation of infrared divegences is known as the **Kinoshita-Lee-Nauenberg theorem**, and in the case of UV divergences what we have seen is part of the **renormalization** which we will discuss more in the following section.



7.4 Photon self energy

A diagram which yields a contribution of the same order in the QED coupling as the previous diagrams is the one in which we draw an electron loop on the photon line:



This is also a virtual correction so it does not change the kinematics. In the Feynman gauge this corresponds to a matrix element,

$$i\mathcal{M}^{\gamma}(p,p') = -ie\left[\overline{u}_{s'}(p')\gamma^{\mu}u_s(p)\right] \times \frac{-ig_{\mu\alpha}}{q^2+i\epsilon}\left[i\Pi^{\alpha\beta}(q)\right]\frac{-ig_{\beta\nu}}{q^2+i\epsilon}\Phi^{\nu}(q)\,,$$

where

$$\begin{aligned} i\Pi^{\alpha\beta}(q) &= -(-ie)^2\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^{\beta}\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\gamma^{\alpha}\frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2+i\epsilon}\right] \\ &= -e^2\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^{\beta}\frac{(\not{k}+m)}{k^2-m^2+i\epsilon}\gamma^{\alpha}\frac{(\not{k}+\not{q}+m)}{(k+q)^2-m^2+i\epsilon}\right] \qquad (7.82) \\ &= -4e^2\int\frac{d^4k}{(2\pi)^4}\frac{k^{\alpha}(k+q)^{\beta}+k^{\beta}(k+q)^{\alpha}-g^{\alpha\beta}\left(k^2+k\cdot q-m^2\right)}{\left[k^2-m^2+i\epsilon\right]\left[(k+q)^2-m^2+i\epsilon\right]}\,. \end{aligned}$$

The overall minus sign comes from the factor (-1) in the context of closed fermion loop. We proceed as in the previous loop calculations and use the

Feynman parametrization:

$$\frac{1}{\left[k^2-m^2+i\epsilon\right]\left[(k+q)^2-m^2+i\epsilon\right]} \tag{7.83}$$

$$= \int_0^1 dx dy \delta\left(1-x-y\right) \frac{1}{\Big[y\left[k^2-m^2+i\epsilon\right]+x\left[(k+q)^2-m^2+i\epsilon\right]\Big]^2}$$

$$= \int_0^1 dx dy \delta\left(1-x-y\right) \frac{1}{\Big[(k^2-m^2+i\epsilon)(x+y)+x\left[2k\cdot q+q^2\right]\Big]^2}$$

$$= \int_0^1 dx \frac{1}{\Big[k^2+2xk\cdot q-m^2+xq^2+i\epsilon\Big]^2}\,.$$

We complete the square, $k^2+2xk\cdot q=(k+xq)^2-x^2q^2$, so that

$$\frac{1}{\left[k^2-m^2+i\epsilon\right]\left[(k+q)^2-m^2+i\epsilon\right]} = \int_0^1 dx \frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^2}\,, \tag{7.84}$$

with

$$\ell = k+xq\,, \tag{7.85}$$

$$\Delta = m^2-x(1-x)q^2>0\,. \tag{7.86}$$

In the numerator of (7.82),

$$k^\alpha(k+q)^\beta+k^\beta(k+q)^\alpha-g^{\alpha\beta}\left(k^2+k\cdot q-m^2\right) \tag{7.87}$$

$$= (\ell-xq)^\alpha((\ell-xq)+q)^\beta+(\ell-xq)^\beta((\ell-xq)+q)^\alpha$$

$$- g^{\alpha\beta}\left((\ell-xq)^2+(\ell-xq)\cdot q-m^2\right)$$

$$\hat{=} 2\ell^\alpha\ell^\beta-g^{\alpha\beta}\ell^2-2x(1-x)q^\alpha q^\beta+g^{\alpha\beta}\left(m^2+x(1-x)q^2\right)\,,$$

where we discarded the terms linear in ℓ. Thus, at this point,

$$i\Pi^{\alpha\beta}(q) = -4e^2\int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^2} \tag{7.88}$$

$$\Big[2\ell^\alpha\ell^\beta-g^{\alpha\beta}\ell^2-2x(1-x)q^\alpha q^\beta+g^{\alpha\beta}\left(m^2+x(1-x)q^2\right)\Big]\,.$$

This is again UV divergent but this time there's no IR divergence since $\Delta > 0$ due to the electron mass. We could use the Pauli-Villars regularization but for fermion loops it's not as convenient as with photon loops. At this point we well shift to the modern **dimensional regularization**.

Dimensional regularization

The idea is super simple: A typical loop integral is of the form,

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 - \Delta + i\epsilon]^2} . \tag{7.89}$$

by Wick's rotation,

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 - \Delta + i\epsilon]^2} = i \int \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{[\ell_E^2 + \Delta]^2} . \tag{7.90}$$

This is clearly infinite,

$$\int \frac{d^4\ell_{\rm E}}{(2\pi)^4} \frac{1}{(\ell_{\rm E}^2 + \Delta)^2} = \int \frac{d\Omega_4}{(2\pi)^4} \int_0^\infty \frac{d|\ell_{\rm E}||\ell_{\rm E}|^3}{(\ell_{\rm E}^2 + \Delta)^2} \sim \log(\infty) . \tag{7.91}$$

If, instead of 4 space-time dimensions, we have d dimensions,

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{[\ell^2 - \Delta + i\epsilon]^2} , \tag{7.92}$$

performing the Wick rotation,

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{[\ell^2 - \Delta + i\epsilon]^2} = i \int \frac{d^d\ell_E}{(2\pi)^d} \frac{1}{[\ell_E^2 + \Delta]^2} , \tag{7.93}$$

we find a finite result:

$$\int \frac{d^d\ell_{\rm E}}{(2\pi)^d} \frac{1}{(\ell_{\rm E}^2 + \Delta)^2} = \int \frac{d\Omega_d}{(2\pi)^d} \int_0^\infty \frac{d|\ell_{\rm E}||\ell_{\rm E}|^{d-1}}{(\ell_{\rm E}^2 + \Delta)^2} < \infty , \text{ if } d < 4. \tag{7.94}$$

Thus, we can regularize the UV divergence by **reducing** the number of space-time dimensions. Also the IR divergence can be regularized by this method but in this case we need to **increase** the number of dimensions.

Sometimes – or actually very often – both are regularized at once by dim.reg. which is bit of a tricky business.

The angular integral in d dimensions goes with a Gaussian integral,

$$(\sqrt{\pi})^d = \left(\int dx e^{-x^2}\right)^d = \int d^d x \exp\left[-\sum_{i=1}^{d} x_i^2\right] \tag{7.95}$$

$$= \int d\Omega_d \int_0^\infty dx x^{d-1} e^{-x^2} ,$$

and making a change of variables $y = x^2$, $dy = 2xdx$,

$$(\sqrt{\pi})^d = \left(\int d\Omega_d\right) \frac{1}{2} \int_0^\infty dy y^{(d/2-1)} e^{-y} .$$

We can identify here the integral representation of the Γ function,

$$\Gamma(z) = \int_0^\infty dx x^{z-1} e^{-x} , \quad \mathrm{Re}\,(z) > 0 \tag{7.96}$$

so

$$(\sqrt{\pi})^d = \left(\int d\Omega_d\right) \frac{1}{2} \Gamma\left(\frac{d}{2}\right) .$$

Thus,

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma\,(d/2)} . \tag{7.97}$$

Also the radial part of (7.94) can be turned into Γ functions:

$$\int_0^\infty \frac{d|\ell_{\mathrm{E}}||\ell_{\mathrm{E}}|^{d-1}}{(|\ell_{\mathrm{E}}|^2 + \Delta)^2} = \frac{1}{2} \int_0^\infty \frac{d|\ell_{\mathrm{E}}|^2 (|\ell_{\mathrm{E}}|^2)^{d/2-1}}{(|\ell_{\mathrm{E}}|^2 + \Delta)^2} \tag{7.98}$$

We do a change of variables, $x = \Delta/(|\ell_{\mathrm{E}}|^2 + \Delta)$, $dx = -d|\ell_{\mathrm{E}}|^2 \Delta/(|\ell_{\mathrm{E}}|^2 +$

$\Delta)^2$, $x : 1 \to 0$,

$$\int_0^\infty \frac{d|\ell_{\rm E}||\ell_{\rm E}|^{d-1}}{(|\ell_{\rm E}|^2+\Delta)^2} = \frac{1}{2}\int_0^1 dx \frac{(|\ell_{\rm E}|^2+\Delta)^2}{\Delta}\frac{(\frac{\Delta}{x}-\Delta)^{d/2-1}}{(|\ell_{\rm E}|^2+\Delta)^2} \tag{7.99}$$

$$= \frac{1}{2}\Delta^{d/2-2}\int_0^1 dx x^{1-d/2}(1-x)^{d/2-1}\,.$$

The remaining x integral matches with the definition of the so-called β function which, in turn, is related to Γ function,

$$B(\alpha,\beta) \equiv \int_0^1 dx x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,, \tag{7.100}$$

when ${\rm Re}\,(\alpha) > 0$ and ${\rm Re}\,(\beta) > 0$. This follows directly from the definition of the Γ function (7.96). By using the above identity,

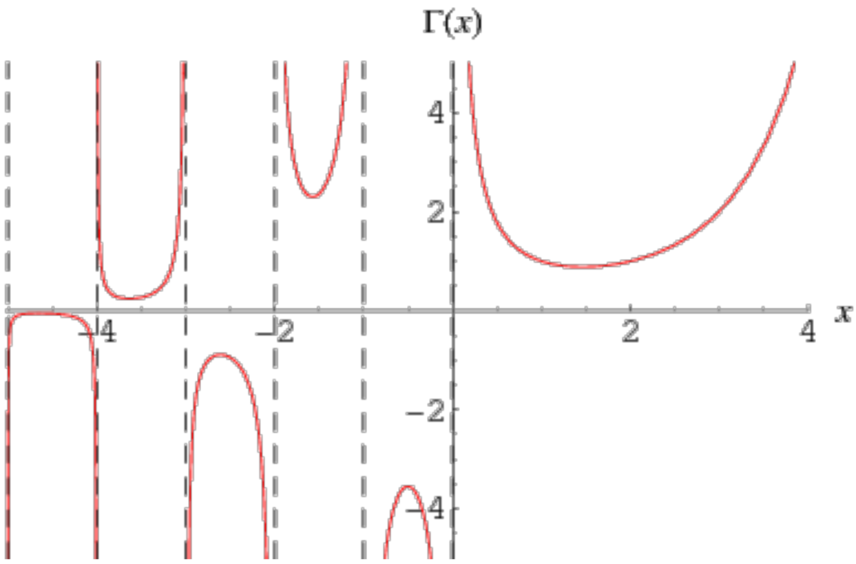
$$\int_0^\infty \frac{d|\ell_{\rm E}||\ell_{\rm E}|^{d-1}}{(|\ell_{\rm E}|^2+\Delta)^2} = \frac{1}{2}\Delta^{\frac{d}{2}-2}\frac{\Gamma(2-d/2)\Gamma(d/2)}{\Gamma(2)}\,.$$

In total,

$$\int \frac{d^d\ell_{\rm E}}{(2\pi)^d}\frac{1}{(\ell_{\rm E}^2+\Delta)^2} = \frac{2\pi^{d/2}}{\Gamma\,(d/2)}\frac{1}{(2\pi)^d}\frac{1}{2}\Delta^{\frac{d}{2}-2}\,\Gamma(2-d/2)\Gamma(d/2) \tag{7.101}$$

$$= \frac{1}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\Gamma(2-d/2)\,, \quad d<4\,.$$

The $\Gamma(z)$ function is singular at $z=0$,



and thus $\Gamma(2-d/2)$ is singular when $d=4$. It is customary to write $d=4-\epsilon$, where $\epsilon>0$, and by using the definition of the Γ function,

$$\Gamma\left(2-\frac{d}{2}\right)=\Gamma\left(\frac{\epsilon}{2}\right)=\frac{2}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon), \tag{7.102}$$

where γ_E is the Euler-Mascheroni constant,

$$\gamma_E\equiv-\int_0^\infty e^{-x}\log x\approx 0.5772\,. \tag{7.103}$$

By using this expansion, we can finally write the singularity structure of the integral (7.101) explicitly,

$$\begin{aligned}\int\frac{d^d\ell_{\mathrm{E}}}{(2\pi)^d}\frac{1}{(\ell_{\mathrm{E}}^2+\Delta)^2}&=\frac{1}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\Gamma(2-d/2) \\ &\overset{\epsilon\to 0}{=}\frac{1}{(4\pi)^2}\left[\frac{2}{\epsilon}-\gamma_E-\log\Delta+\log(4\pi)\right]\,.\end{aligned} \tag{7.104}$$

We see that the logarithmic UV divergence corresponds in dimensional regulariztion to $1/\epsilon$ pole. It should be born in mind that **the parameter ϵ appearing here has nothing to do with the ϵ that appears in the propagators!**

With a bit of tinkering, one can verify the following general identities,

$$\int\frac{d^N\ell}{(2\pi)^N}\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^m}=\frac{i(-1)^m}{(4\pi)^{N/2}}\frac{\Gamma\left(m-N/2\right)}{\Gamma\left(m\right)}\left(\frac{1}{\Delta}\right)^{m-N/2} \tag{7.105}$$

$$\int\frac{d^N\ell}{(2\pi)^N}\frac{\ell^2}{\left[\ell^2-\Delta+i\epsilon\right]^m}=\frac{-i(-1)^m}{(4\pi)^{N/2}}\frac{N}{2}\frac{\Gamma\left(m-N/2-1\right)}{\Gamma\left(m\right)}\left(\frac{1}{\Delta}\right)^{m-N/2-1}$$

When the dimension of the space time is N, the energy-momentum vectors are of the form,

$$p^\mu=(p^0,p^1,p^2,\ldots,p^{N-1}), \tag{7.106}$$

and thus also the indices of the metric tensor $g^{\mu\nu}$ run from 0 to $N-1$,

$$g^{\mu\nu}g_{\mu\nu}=N\,. \tag{7.107}$$

For this reason also the γ-matrix algebra slightly changes. This is not unique, but usually the following identities are retained intact,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,, \quad \mathrm{Tr}\;(I) = 4\,, \tag{7.108}$$

and it follows that

$$\gamma^\mu \gamma^\nu \gamma_\mu = -(N-2)\gamma^\nu \tag{7.109}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4g^{\rho\nu} + (N-4)\gamma^\nu\gamma^\rho \tag{7.110}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-N)\gamma^\nu\gamma^\rho\gamma^\sigma \tag{7.111}$$

$$\int \frac{d^N\ell}{(2\pi)^N}\frac{\ell^\mu\ell^\nu}{D(\ell^2)} = \frac{1}{N}g^{\mu\nu}\int \frac{d^N\ell}{(2\pi)^N}\frac{\ell^2}{D(\ell^2)}\,. \tag{7.112}$$

Lastly, the QED coupling becomes dimensionful quantity. Since the action,

$$S = \int d^4x \mathcal{L}_{\mathrm{QED}} \tag{7.113}$$

is dimensionless, in 4 dimensions we have $\dim[\mathcal{L}_{\mathrm{QED}}] = 4$ (in dimensions of mass). The QED Lagrangian density was,

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi - e\overline{\psi}\gamma^\mu\psi A_\mu\,, \tag{7.114}$$

so we can infer,

$$\dim[\psi] = 3/2\,, \tag{7.115}$$

$$\dim[A] = 1\,, \tag{7.116}$$

$$\dim[e] = 0\,. \tag{7.117}$$

When the space-time dimension is N, we have $\dim[\mathcal{L}^N_{\mathrm{QED}}] = N$, and

$$\dim[\psi] = (N-1)/2\,, \tag{7.118}$$

$$\dim[A] = N/2 - 1\,, \tag{7.119}$$

$$\dim[e] = 2 - N/2\,. \tag{7.120}$$

When $N = 4-\epsilon$, then $\dim[e] = \epsilon/2$. Often, the dimension of the spacetime is written explicitly using an arbitrary mass scale μ_D as,

$$e \to e\mu_D^{2-N/2}. \tag{7.121}$$

Let's now continue with the photon self-energy diagram from Eq. (7.88), but now in N dimensions,

$$i\Pi^{\alpha\beta}(q) = -4e^2\mu_D^{4-N}\int_0^1 dx\int\frac{d^N\ell}{(2\pi)^N}\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^2} \tag{7.122}$$

$$\left[2\ell^\alpha\ell^\beta - g^{\alpha\beta}\ell^2 - 2x(1-x)q^\alpha q^\beta + g^{\alpha\beta}\left(m^2+x(1-x)q^2\right)\right]$$

$$= -4e^2\mu_D^{4-N}\int_0^1 dx\int\frac{d^N\ell}{(2\pi)^N}\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^2}$$

$$\left[(2/N-1)g^{\alpha\beta}\ell^2 - 2x(1-x)q^\alpha q^\beta + g^{\alpha\beta}\left(m^2+x(1-x)q^2\right)\right].$$

The required ℓ integrals are,

- $$\int\frac{d^N\ell}{(2\pi)^N}\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^2} = \frac{i}{(4\pi)^{N/2}}\Gamma\left(2-N/2\right)\left(\frac{1}{\Delta}\right)^{2-N/2} \tag{7.123}$$

- $$\int\frac{d^N\ell}{(2\pi)^N}\frac{(2/N-1)\ell^2}{\left[\ell^2-\Delta+i\epsilon\right]^2} = \frac{-i}{(4\pi)^{N/2}}\frac{N}{2}(2/N-1)\Gamma\left(1-N/2\right)\left(\frac{1}{\Delta}\right)^{2-N/2-1}$$

$$= \frac{-i}{(4\pi)^{N/2}}(1-\frac{N}{2})\Gamma\left(1-N/2\right)\left(\frac{1}{\Delta}\right)^{2-N/2-1}$$

$$= \frac{-i}{(4\pi)^{N/2}}\Gamma\left(2-N/2\right)\left(\frac{1}{\Delta}\right)^{2-N/2-1}.$$

Using these, we get,

$$i\Pi^{\alpha\beta}(q) = \tag{7.124}$$

$$= -4e^2\mu_D^{4-N}\int_0^1 dx\left[\frac{i}{(4\pi)^{N/2}}\left(\frac{1}{\Delta}\right)^{2-N/2}\Gamma\left(2-N/2\right)\right]$$

$$\left[(-m^2+x(1-x)q^2)g^{\alpha\beta} - 2x(1-x)q^\alpha q^\beta + g^{\alpha\beta}\left(m^2+x(1-x)q^2\right)\right].$$

The lowest line simplifies to

$$2x(1-x)\left[q^2 g^{\alpha\beta} - q^\alpha q^\beta\right], \tag{7.125}$$

so finally,

$$i\Pi^{\alpha\beta}(q) = \left[q^2 g^{\alpha\beta} - q^\alpha q^\beta\right] \times i\Pi(q^2) \tag{7.126}$$

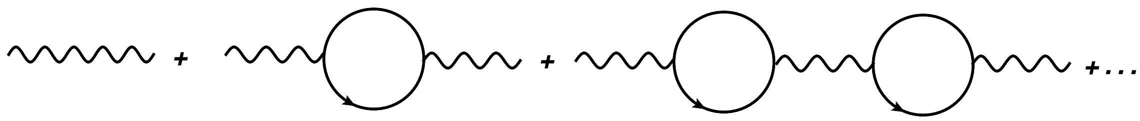
$$i\Pi(q^2) = \frac{-8ie^2\mu_D^{4-N}}{(4\pi)^{N/2}}\int_0^1 dx x(1-x)\left[\left(\frac{1}{\Delta}\right)^{2-N/2}\Gamma\left(2-N/2\right)\right]$$

$$\stackrel{\epsilon\to 0}{=} \frac{-2i\alpha}{\pi}\int_0^1 dx x(1-x)\left[\frac{2}{\epsilon} - \gamma_E + \log\frac{\mu_D^2}{\Delta} + \log(4\pi)\right]$$

We note that $\Pi^{\alpha\beta}(q)$ fulfills the Ward identity,

$$q_\alpha \Pi^{\alpha\beta}(q) = q_\beta \Pi^{\alpha\beta}(q) = 0\,, \tag{7.127}$$

as we might have expected. We proceed as in the electron self-energy calculation and sum the obtained result to all orders,



This corresponds to,

$$\frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\alpha}}{q^2}i\Pi^{\alpha\beta}(q)\frac{-ig_{\beta\nu}}{q^2} + \frac{-ig_{\mu\alpha}}{q^2}i\Pi^{\alpha\beta}(q)\frac{-ig_{\beta\rho}}{q^2}i\Pi^{\rho\sigma}(q)\frac{-ig_{\sigma\nu}}{q^2} + \cdots$$

and after some small tinkering,

$$\frac{-i}{q^2\left[1-\Pi(q)\right]}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)-i\left(\frac{q_\mu q_\nu}{q^4}\right)\,. \tag{7.128}$$

Those terms which are proportional to $q_\mu q_\nu$ will, according to the Ward identity, yield zero in scattering amplitudes so only the $g_{\mu\nu}$ term is relevant. The full propagator thus reads,

$$\frac{-ig_{\mu\nu}}{q^2\left[1-\Pi(q)\right]}\,. \tag{7.129}$$

The summed propagator clearly has a pole at $q^2 = 0$ so the **photon remains massless**. Close to the pole the propagator behaves, obviously, as

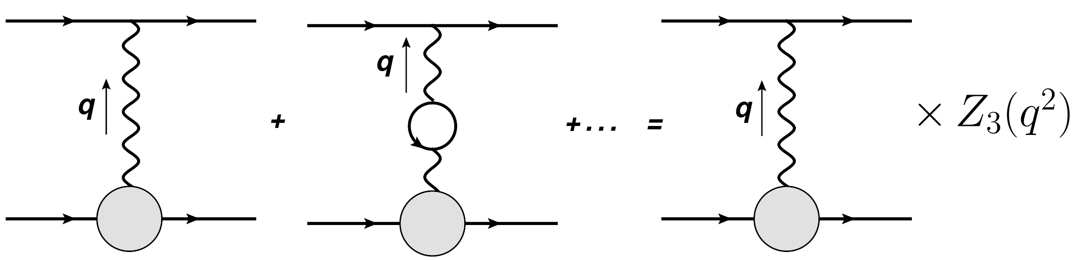
$$\frac{-ig_{\mu\nu}Z_3}{q^2}\,, \tag{7.130}$$

where Z_3 is the renormalization constant related to the photon field,

$$Z_3 = \frac{1}{\left[1-\Pi(0)\right]} = 1-\frac{\alpha}{3\pi}\left[\frac{2}{\epsilon}-\gamma_E+\log\frac{\mu_D^2}{m^2}+\log(4\pi)\right]\,. \tag{7.131}$$

This is what we would use (according to the LSZ theorem) if our scattering amplitude contains external photons.

Now we don't have external photons in the game, but the virtual electron loop yields a multiplicative factor $Z_3(q^2) \equiv 1/\left[1-\Pi(q^2)\right]$:



So where should we stuff the UV-divergence that $Z_3(q^2)$ entails? In analogy to the mass renormalization, this infinity is absorbed into a redefinition of the electric charge — **charge renormalization**. We now denote the charge that appears in the original Lagrangian by e_0 and call it the **bare charge**. Since an internal photon propagator always starts and ends to a vertex factor $-ie_0\gamma^\mu$, it is natural to share the contribution of $Z_3(q^2)$ evenly with both. In addition, as $Z_3(q^2)$ depends on the scale q^2, we define an **effective charge/coupling** or **running charge/coupling**,

$$e_{\text{eff}}(q^2) \equiv e_0\sqrt{Z_3(q^2)}\,, \tag{7.132}$$

or in terms of the fine-structure constant $\alpha = e^2/4\pi$,

$$\alpha_{\text{eff}}(q^2) \equiv \alpha_0\, Z_3(q^2)\,. \tag{7.133}$$

This would indicate that the measured charge will depend on a scale (momentum transfer). The charge that an experimentalist will measure is definitely a finite number, so because $Z_3(q^2)$ is infinite, also the bare charge α_0 has to be infinite as well.

The effective coupling $\alpha_{\text{eff}}(q^2)$ thus depends on the scale. How? According to the definition,

$$\alpha_{\text{eff}}(q^2) = \frac{\alpha_0}{1-\Pi(q^2)}\,, \tag{7.134}$$

so

$$\frac{1}{\alpha_{\text{eff}}(q^2)} = \frac{1}{\alpha_0} - \frac{\Pi(q^2)}{\alpha_0}\,. \tag{7.135}$$

The low-energy measurement give $\alpha \equiv \alpha_{\text{eff}}(0) \approx 1/137$, so we use this as a reference value,

$$\begin{aligned}\frac{1}{\alpha_{\text{eff}}(q^2)} &= \frac{1}{\alpha_0} - \frac{\Pi(0)}{\alpha_0} + \frac{\Pi(0)}{\alpha_0} - \frac{\Pi(q^2)}{\alpha_0} \\ &= \frac{1}{\alpha} - \frac{1}{\alpha_0}\left[\Pi(q^2) - \Pi(0)\right]\end{aligned} \tag{7.136}$$

According to Eq. (7.126),

$$\begin{aligned}\Pi(q^2) - \Pi(0) &= \frac{-2\alpha_0}{\pi}\int_0^1 dx x(1-x)\log\frac{m^2}{m^2 - x(1-x)q^2} \\ &\xrightarrow{-q^2 \gg m^2} \frac{\alpha_0}{3\pi}\left[\log\left(\frac{-q^2}{m^2}\right) - \frac{5}{3}\right]\,,\end{aligned} \tag{7.137}$$

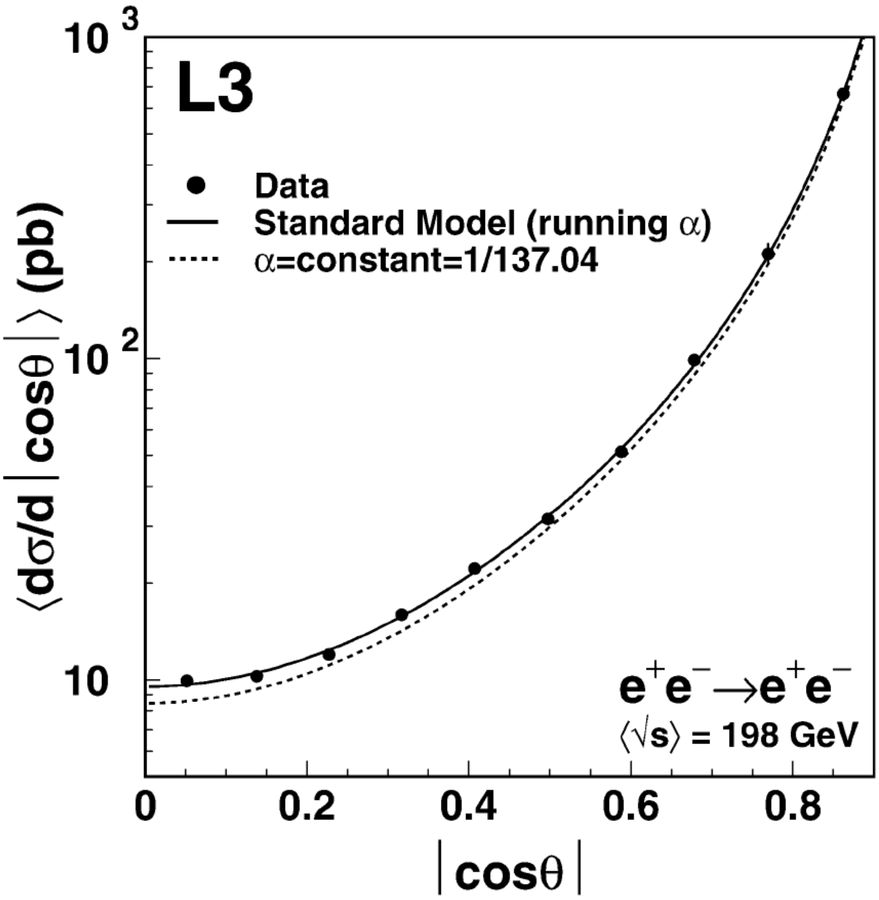
so

$$\frac{1}{\alpha_{\rm eff}(q^2)} = \frac{1}{\alpha} - \frac{1}{3\pi}\left[\log\left(\frac{-q^2}{m^2}\right) - \frac{5}{3}\right]. \tag{7.138}$$

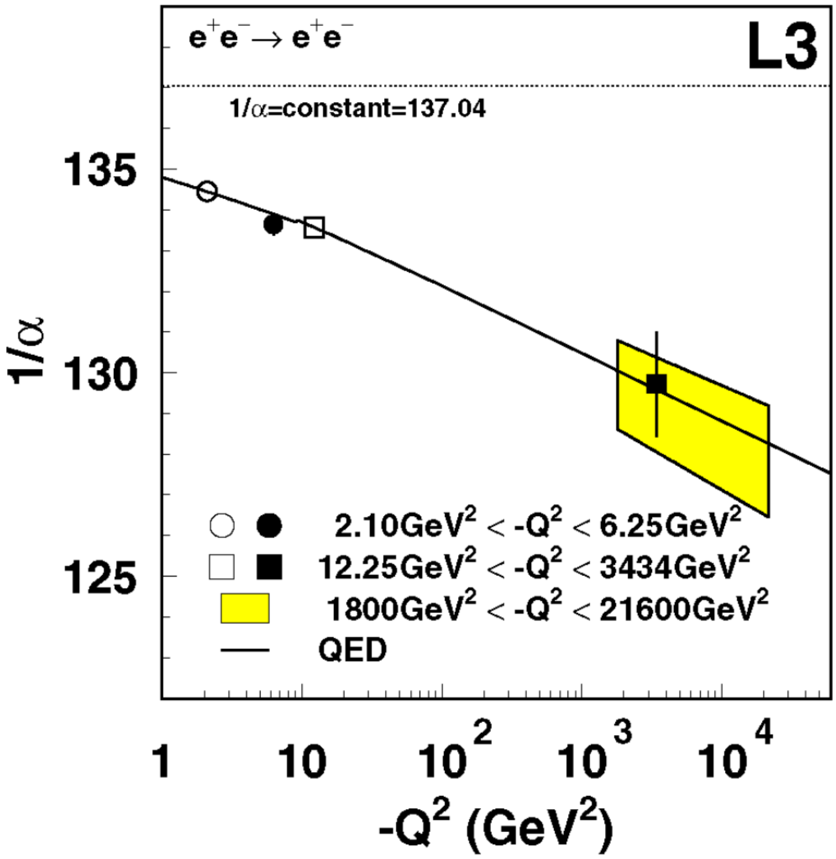
This gives the final form of the scale-dependent coupling (to first order),

$$\alpha_{\rm eff}(q^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\log\left(\frac{-q^2}{m^2}\right)}, \quad -q^2 \gg m^2. \tag{7.139}$$

When $-q^2$ grows, the denominator of the equation above diminishes, so the coupling becomes stronger. The change is relatively slow (logarithmic) but it has been verified experimentally. Below we show some result from the LEP collider for the angular dependence in $e^+e^- \to e^+e^-$ process [Phys.Lett. B623 (2005) 26-36].



Without a scale-dependent coupling the shape of the theoretical curve deviates from the measurements. Accounting for the scale dependence in coupling even visually improves the correspondence. Below still the extracted $\alpha_{\text{eff}}(q^2)$.



The measurements thus clearly prefer the scale dependence of the coupling constant.

One can, of course, always express the physical cross sections also in terms of scale-independent coupling e.g. $\alpha = \alpha_{\text{eff}}(q^2 = 0) \approx 1/137$ which also removes the $1/\epsilon$ poles and dependence of the unphysical parameter μ_D^2 perfectly fine. However, in this case our expression for the cross section would explicitly involve powers of logarithms of the form $\alpha \log(-q^2/m^2)$ which can be large if $-q^2 \gg m^2$ and thereby worsen the convergence of the perturbative series. By expressing the cross sections in terms of running coupling $\alpha_{\text{eff}}(q^2)$ effectively resums these logarithms into the definition of the coupling stabilizing the perturbative series. The fact that $\alpha_{\text{eff}}(q^2)$ resums such logarithms to all orders can be seen also by expanding Eq. (7.139) in powers of α.

The scale dependence or **running** of the coupling is often expressed in terms of the so-called **β function**,

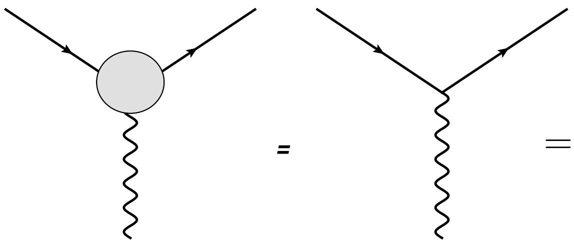
$$\beta(Q^2) \equiv Q^2 \, \frac{d\alpha_{\text{eff}}(Q^2)}{dQ^2}\,, \qquad Q^2 \equiv -q^2\,. \tag{7.140}$$

From Eq. (7.139) we can easily check that for QED (to lowest order),

$$\beta(Q^2) = \frac{\alpha_{\text{eff}}^2(Q^2)}{3\pi}\,, \quad Q^2 \gg m^2\,. \tag{7.141}$$

This also clearly shows that the coupling constant monotonically increases as the scale Q^2 grows.

The fact that the behaviour of QED coupling $\alpha_{\text{eff}}(Q^2)$ is completely dictated by the photons self energy diagram is not general but is specific to QED. Let us denote the UV-divergent part of the loop-corrected vertex by $1/\tilde{Z}_1$,



$$= -ie\gamma^\mu \times \tilde{Z}_1^{-1}(q^2)$$

According to Eq. (7.56), with Pauli-Villars regularization,

$$\tilde{Z}_1^{-1}(q^2) = 1 + \frac{\alpha_0}{2\pi}\left[\frac{1}{2}\log\left(\frac{\Lambda^2}{-q^2}\right) + \cdots\right].$$

or the same in dimensional regularization (Ex.),

$$\tilde{Z}_1^{-1}(q^2) = 1 + \frac{\alpha_0}{2\pi}\frac{1}{2}\left[\frac{2}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu_D^2}{-q^2}\right) + \cdots\right]. \tag{7.142}$$

We then denote the UV-divergent part of the electron self-energy (after mass renormalization) by $\tilde{Z}_2$. According to Eq. (7.76), with Pauli-Villars regularization,

$$\tilde{Z}_2(q^2) = 1 - \frac{\alpha}{2\pi}\left[\frac{1}{2}\log\left(\frac{\Lambda^2}{-q^2}\right) + \cdots\right]$$

which in dimensional regularization corresponds to (Ex.),

$$\tilde{Z}_2(q^2) = 1 - \frac{\alpha_0}{2\pi}\frac{1}{2}\left[\frac{2}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu_D^2}{-q^2}\right) + \cdots\right] . \tag{7.143}$$

Both external electrons contribute by $\sqrt{\tilde{Z}_2}$. Finally, we denote by $\tilde{Z}_3(q^2)$ the UV-divergent part of the photon self-energy correction,

$$\tilde{Z}_3(q^2) = 1 - \frac{\alpha_0}{3\pi}\left[\frac{2}{\epsilon} + \log\left(\frac{\mu_D^2}{-q^2}\right) - \gamma_E + \log(4\pi)\cdots\right]$$

In general we should define the scale-dependent coupling by

$$e_{\text{eff}}(q^2) \equiv e_0\frac{\tilde{Z}_2(q^2)\sqrt{\tilde{Z}_3(q^2)}}{\tilde{Z}_1(q^2)} , \tag{7.144}$$

but in QED it so happens that $\tilde{Z}_2(q^2)/\tilde{Z}_1(q^2)$ is not UV divergent so only the photon self-energy correction is enough to renormalize the QED coupling. In other theories (e.g. QCD), this may not be the case and all the legs i connecting to a given vertex will give one $\sqrt{\tilde{Z}_i}$ and the vertex-correction itself one $\tilde{Z}_1^{-1}$.

Schemes and scales

What terms to include into the renormalizaton factors $\tilde{Z}_i(q^2)$ when defining the running coupling constant by Eq. (7.144) is not unique. Different choices are called **renormalization schemes**. In dimensional regularization by far the most common is the so-called **modified minimal subtraction**

scheme or just $\overline{\mathrm{MS}}$ **scheme** in short. In this scheme one defines,

$$\tilde{Z}_3(q^2) \overset{\overline{\mathrm{MS}}}{=} 1 - \frac{\alpha_0}{3\pi}\left[\frac{2}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu_D^2}{-q^2}\right)\right], \tag{7.145}$$

$$\tilde{Z}_2(q^2) \overset{\overline{\mathrm{MS}}}{=} 1 - \frac{\alpha_0}{2\pi}\frac{1}{2}\left[\frac{2}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu_D^2}{-q^2}\right)\right], \tag{7.146}$$

$$\tilde{Z}_1^{-1}(q^2) \overset{\overline{\mathrm{MS}}}{=} 1 + \frac{\alpha_0}{2\pi}\frac{1}{2}\left[\frac{2}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu_D^2}{-q^2}\right)\right]. \tag{7.147}$$

so the definition absorbs not only the $1/\epsilon$ pole but also factors γ_E ja $\log(4\pi)$ typical to the dimensional regularization. In the so-called **minimal subtraction scheme** or MS **scheme** in short, these terms are left out from the definition.

To some extent, the choice of scheme affects e.g. what kind of β function we get. At least the first five terms of the QED β function have been calculated. In the $\overline{\mathrm{MS}}$ scheme the first three terms are,

$$\beta(Q^2) = \frac{\alpha_{\mathrm{eff}}^2(Q^2)}{3\pi} + \frac{\alpha_{\mathrm{eff}}^3(Q^2)}{4\pi^2} - \frac{31\alpha_{\mathrm{eff}}^4(Q^2)}{288\pi^3}. \tag{7.148}$$

Another ambiguity is related to the scale q^2. As we see from the definition (7.144), we can express e_0 in terms of whatever scale q^2. It is natural to tie this scale to some invariant scale that appears in the process but there is no single correct way to choose this. The chosen scale is called the **renormalization scale**.

In a physical observable, two different renormalization schemes or scale choices formally differ by a factor that is higher order in coupling than the precision of the calculation. In this sense all schemes and scales are equally good. Numerically they are not exactly equal, though. By performing the calculation in more than one scheme and with several scale choices serves as a tool to test the perturbative reliability of the result.