Problem set 9 Return by Wed 20.3. at 16.00 to the box labeled FYST530

1. Consider two unit vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{v}}$, specified by angles $(\theta_{\hat{\mathbf{n}}}, \varphi_{\hat{\mathbf{n}}})$ and $(\theta_{\hat{\mathbf{v}}}, \varphi_{\hat{\mathbf{v}}})$, and let γ be the angle between them i.e $\hat{\mathbf{n}} \cdot \hat{\mathbf{v}} = \cos\gamma$. Prove the **Addition theorem for spherical harmonics**:

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\hat{\mathbf{n}}) Y_{\ell m}(\hat{\mathbf{v}}),$$

where $P_\ell(x)$ denotes the Legendre polynomial, and $Y_{\ell m}$s are standard spherical harmonics.

Instructions: Begin by considering an arbitrary rotation R which takes $\hat{\mathbf{n}} \xrightarrow{R} \hat{\mathbf{n}}'$, $\hat{\mathbf{v}} \xrightarrow{R} \hat{\mathbf{v}}'$, and note that ($\equiv$ derive)

$$Y_{\ell m}(\hat{\mathbf{n}}') = \sum_{m'} Y_{\ell m'}(\hat{\mathbf{n}}) D^{(\ell)\,*}_{mm'}(R),$$

where $D^{(\ell)\,*}_{mm'}(R)$s are Wigner's functions. From this you should be able to derive a relation

$$\sum_m Y^*_{\ell m}(\hat{\mathbf{n}}) Y_{\ell m}(\hat{\mathbf{v}}) = \sum_m Y^*_{\ell m}(\hat{\mathbf{n}}') Y_{\ell m}(\hat{\mathbf{v}}'),$$

which reduces to the desired theorem with suitably chosen R.

2. As suggested in the lecture notes p.283, find the $SU(2)$ rotation group $j = 1$ representation matrix $\mathbb{D}^{(1)}(\alpha, \beta, \gamma)$.

3. (a) Let's check the $\alpha \leftrightarrow \gamma$ typo in the 1st box on p. 283 (the right hand sides of the equations):
   Using the p. 264 result of the correspondence between $SO(3)$ and $SU(2)$ group elements and using the general rotation matrix $D^{\frac{1}{2}}$ which we obtaned on p. 282, show that the $\alpha \leftrightarrow \gamma$-corrected form of $R(\alpha, \beta, \gamma)$ indeed follows.
   (b) Calculate the Clebsch-Gordan coefficients ${}_u\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle_c$ for

$$j_1 = 3 \quad j_2 = \frac{3}{2} \quad j = \frac{5}{2} \quad m = \frac{5}{2}$$

4. With the aid of Gaunt's formula, restudy the problem about selection rules for electric dipole transitions in a Hydrogen-like atom, which we considered earlier (p.174-183) in the lectures:
   (a) Write down Gaunt's formula using the $3j$-symbols. Find the selection rules corresponding to the integral

$$\int d\Omega\, Y^*_{l_f m_f}(\Omega)\, Y_{1m}(\Omega)\, Y_{l_i m_i}(\Omega).$$

   (b) Solve this integral for $m = 0, \pm 1$ for all allowed values of l_f and l_i. Make use of properties of the $3j$-symbols presented on p. 299 and 300.