Problem set 5 Return by Wed 20.2. at 16.00 to the box labeled FYST530

1. The Hamiltonian operator for the harmonic oscillator is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

Using the commutation relations between position and momentum operators, write down, and solve the equations of motion

$$\frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t} = \frac{i}{\hbar}[\hat{H}, \hat{x}(t)] \qquad \frac{\mathrm{d}\hat{p}(t)}{\mathrm{d}t} = \frac{i}{\hbar}[\hat{H}, \hat{p}(t)]$$

for the Heisenberg picture operators $\hat{x}(t)$ and $\hat{p}(t)$ in terms of Schrdinger picture operators $\hat{x} = \hat{x}(t = 0)$ and $\hat{p} = \hat{p}(t = 0)$.

2. (a) Show that the Lippman-Schwinger equation

$$|\Psi_a\rangle = |\Phi_a\rangle + \frac{1}{E_a - \hat{H}_0 + i\epsilon}\hat{V}_S|\Psi_a\rangle$$

is equivalent to the coordinate space integral equation, encountered in the context of potential scattering,

$$\Psi_{\mathbf{k}}(\mathbf{r}) = \Phi_{\mathbf{k}}(\mathbf{r}) + \frac{2\mu}{\hbar^2}\int d^3r' G_{\mathbf{k}}(\mathbf{r} - \mathbf{r}')V(\mathbf{r}')\Psi_{\mathbf{k}}(\mathbf{r}').$$

where $G_{\mathbf{k}}(\mathbf{r})$ is the Green's function

$$G_{\mathbf{k}}(\mathbf{r}) = -\int \frac{d^3q}{(2\pi)^3}\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2 - \mathbf{k}^2 - i\epsilon}.$$

In particular, show that

$$\left\langle \mathbf{r} \left| \frac{1}{E_a - \hat{H}_0 + i\varepsilon} \right| \mathbf{r}' \right\rangle = \frac{2\mu}{\hbar^2}G_{\mathbf{k}}(\mathbf{r} - \mathbf{r}').$$

(b) As suggested in the lecture notes p. 130, prove that the scattering amplitude is obtained from the transition matrix elements T_{fi} as

$$f_{\mathbf{k}}(\Omega) = -2\pi^2\frac{2\mu}{\hbar^2}T_{fi}.$$

3. (a) Let's consider a simple two-level system with energy eigenstates $\{|\phi_1\rangle, |\phi_2\rangle\}$, which is perturbed by a potential oscillating with angular frequency ω:

$$V(t) = \begin{pmatrix} 0 & V_{12} \\ V_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \gamma e^{i\omega t} \\ \gamma e^{-i\omega t} & 0 \end{pmatrix},$$

where $\gamma \in \mathbb{R}$. Let the system be initially at state $|\Psi(t = 0)\rangle = |\phi_1\rangle$. Solve the time-evolution of the system exactly and show that the probability of finding the system at later times t in a state $|\phi_2\rangle$ is given by the **Rabi formula**

$$P_2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \delta\omega)^2/4} \sin^2\left(t\sqrt{\gamma^2/\hbar^2 + (\omega-\delta\omega)^2/4}\right),$$

where $\delta\omega \equiv (E_2 - E_1)/\hbar$.

(b) Compute the corresponding transition probabilities using first order time-dependent perturbation theory and compare the results by expanding the exact result in powers of γ, when $\gamma^2 \ll \hbar^2(\omega - \delta\omega)^2/4$. What happens at *resonance frequency* $\omega = \delta\omega$?

4. If two operators do not commute, they cannot be freely moved across each other as ordinary numbers, but there is an extra term from the commutator: $\hat{A}\hat{B} = \hat{B}\hat{A} - [\hat{B}, \hat{A}]$. The situation is similar, but more tricky, in the case of exponentials of operators.

(a) Let $\hat{A}$ and $\hat{B}$ be operators. Consider a function $F(\lambda)$ defined by

$$F(\lambda) = e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}},$$

where λ is a parameter. Show that $F(\lambda)$ obeys

$$\frac{\mathrm{d}F(\lambda)}{\mathrm{d}\lambda} = [\hat{A}, F(\lambda)],$$

and then derive the expression

$$F(\lambda) = B + \frac{\lambda}{1!}[\hat{A}, \hat{B}] + \frac{\lambda^2}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \cdots.$$

(b) Assume now that operators $\hat{A}$ and $\hat{B}$ commute with their commutator: $[A, [A, B]] = [B, [A, B]] = 0$, which would be the case if $\hat{A}$ and $\hat{B}$ were, for example, momentum and position operators. Write down the differential equation obeyed by the operator $\hat{G}(\lambda)$ defined by

$$e^{\lambda\hat{A}}e^{\lambda\hat{B}} = \hat{G}(\lambda)\, e^{\lambda\hat{B}}e^{\lambda\hat{A}}$$

and derive the identity:

$$e^{\hat{A}}e^{\hat{B}} = e^{[\hat{A},\hat{B}]}e^{\hat{B}}e^{\hat{A}}.$$

Under the same assumptions, show that

$$e^{\hat{A}+\hat{B}} = e^{-\frac{1}{2}[\hat{A},\hat{B}]}e^{\hat{A}}e^{\hat{B}}.$$