Problem set 13 Return by $\approx$ **Mon 29.4.** to the box labeled FYST530

1. As suggested in the lecture notes on p. 449, show that

$$\vec{\sigma}\cdot\hat{\mathbf{e}}_{\mathbf{r}}\,\mathcal{Y}_{l_l jm} = -\mathcal{Y}_{l_u jm} \tag{1}$$
$$\vec{\sigma}\cdot\hat{\mathbf{e}}_{\mathbf{r}}\,\mathcal{Y}_{l_u jm} = -\mathcal{Y}_{l_l jm} \tag{2}$$

Hints: Show first that $\{\hat{k}, \vec{\sigma}\cdot\hat{\mathbf{e}}_{\mathbf{r}}\} = 0$, and apply this to $\mathcal{Y}_{l_u jm}$ and $\mathcal{Y}_{l_l jm}$ to deduce the effect of $\vec{\sigma}\cdot\hat{\mathbf{e}}_{\mathbf{r}}$. Then study the effect of $(\vec{\sigma}\cdot\hat{\mathbf{e}}_{\mathbf{r}})^2$ on $\mathcal{Y}_{l_u jm}$ and $\mathcal{Y}_{l_l jm}$ and make also use of the normalization integrals of the $\mathcal{Y}_{ljm}$ to get constraints for the remaining proportionality constants. Finally, to fix the minus signs above, consider a special case $\mathbf{r} = z\hat{e}_z$, (setting $\theta = 0$ in the $\mathcal{Y}_{ljm}$) and apply p. 28 result for the spherical harmonics.

2. Starting from the radial Dirac equations for the hydrogen atom, which we derived at the lectures (p. 450, first box), derive p. 454 result for the exact energy levels of the relativistic hydrogen atom. Do also the series expansion suggested on p. 454 and derive the fine structure of the hydrogen atom (p. 455, 2nd box).

3. *(From this problem you earn* **6** *points)*
Let's consider the Dirac equation in the presence of a radial mass-like scalar potential,

$$(i\hbar\gamma^{\mu}\partial_{\mu} - V(r) - mc) = 0,$$

where we assume the potential to be a box,

$$V(r) = \begin{cases} 0, & r < R \\ V_0 > 0, & r \geq R \end{cases}$$

   (a) Derive (in sufficient detail, using also the result of the Problem 1 above) the radial Dirac equations in this case. The separation of the angular parts (spherical spinors) goes again with the ansatz given on p. 448.

   (b) Consider then massless particles ($m = 0$), find out the positive-energy eigenspinors for a finite V_0 and, requiring continuity of the eigenspinor, write down the transcendent equation from which one can determine the eigenenergies of the system. Determine the energy of the ground state [$n(l_A)_j = 1s_{1/2}$; you should figure out what n now stands for] in the case of an infinitely tall box ($V_0 \to \infty$).

Hints: The result inside the box should be

$$\Psi_{E,j,s,\kappa,m}(\mathbf{r}) = A\begin{pmatrix} j_{l_A}(Er)\,\mathcal{Y}_{l_A jm}(\Omega) \\ i\delta j_{l_A-\delta}(Er)\,\mathcal{Y}_{l_B jm}(\Omega)\end{pmatrix}$$

where the radial functions are spherical Bessel functions, and $\delta \equiv \frac{\kappa}{|\kappa|}$, $\kappa = \mp(j+\frac{1}{2})$, $l_A = j \mp \frac{1}{2}$, $l_B = j \pm \frac{1}{2} = l_A - \delta$, while outside the box the result should be

$$\Psi_{E,j,s,\kappa,m}(\mathbf{r}) = C\begin{pmatrix} k_{l_A}(\sqrt{V_0^2 - E^2}\,r)\,\mathcal{Y}_{l_A jm}(\Omega) \\ -i\sqrt{\frac{V_0-E}{V_0+E}}\,k_{l_A-\delta}(\sqrt{V_0^2-E^2}\,r)\,\mathcal{Y}_{l_B jm}(\Omega)\end{pmatrix}$$

where the radial functions are now modified spherical Bessel functions. Consult the tables (e.g. Abramowitz-Stegun) for the needed recursion relations.

[Historically, this was P. N. Bogoljubov's model (1967) for the quark confinement in hadrons: massless non-interacting spin-half quarks moving in such an infinitely tall radial potential box. A nucleon consists of three quarks in the lowest $1s_{1/2}$ state and each quark has a different color, so that the Pauli exclusion principle is not violated. Setting one of the quarks into the state $2s_{1/2}$, this model was able to predict the so-called Roper resonance $N(1440)$.]

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4. **There hasn't been enough work yet in this course (right?), so there is one extra problem for you. Do it and earn a few extra points!**
   Solve the Pauli Equation for a spin-$\frac{1}{2}$ particle (p. 432) in a constant magnetic field, which points into the z-direction. Find in particular the eigenvalues and eigenfunctions of energy. Let's assume that the scalar potential is zero and the vector potential points into the z-direction. (Hint: use separation of variables).