Problem set 9 Return before 16 on Wednesday 21.3. to the box labeled FYST530

1. In order to gain some further confidence on path integral formulation of quantum dynamics, we'll now derive the Schrödinger equation with the path integral as a starting point.

   In its discretized form, as the time interval $t'' - t$ is split to N pieces of length δt, the propagator $G(x'', t''; x, t)$ consists of x-space integrals at each intermediate time. Singling out the very last time slice, we have

$$\begin{aligned}\Psi(x'', t'') &= \int_{-\infty}^{\infty} \mathrm{d}x\, G(x'', t''; x, t)\, \Psi(x, t) \\ &= \lim_{\delta t \to 0} \sqrt{\frac{m}{2\pi i \hbar \delta t}} \int_{-\infty}^{\infty} \mathrm{d}x_{N-1}\, e^{\frac{i}{\hbar}\delta t \left[\frac{m}{2}\left(\frac{x_N - x_{N-1}}{\delta t}\right)^2 - V(x_{N-1})\right]}\, \Psi(x_{N-1}, t'' - \delta t).\end{aligned}$$

   By expanding the suitable terms in the right-hand side of the equation above around (x'', t''), derive the time-dependent Schrödinger equation.

   Comment: Since $G(x'', t''; x, t)$ is rather a distribution than an ordinary function, you should not be surprised if you encounter non-convergent integrals. These can be handeled by introducing a convergence factor like $-i\epsilon$. Alternatively, you may also think the time as a complex variable, i.e put $t \to -it$ which renders the integrals above real, and well-behaved. At the end you should make an analytical continuation back to real-time by replacing $t \to it$.

2. (a) Show that the orthogonal $N \times N$ matrices O, whose $\det O = +1$, form a group.

   (b) Show that the unitary $N \times N$ matrices U, whose $\det U = +1$, form a group.

   (c) Starting from a general form of a 2×2 -matrix $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that any $SU(2)$ matrix can be written in a form

$$U = a_0 \mathbb{I}_2 + i\mathbf{a} \cdot \sigma,$$

   where σ_is are Pauli matrices and $a_0^2 + \mathbf{a}^2 = 1$.

3. The rotation matrix that induces a rotation by an angle α about an axis $\mathbf{n}$ in ordinary 3-D space, can be written as

$$R_{\mathbf{n}}(\alpha) = e^{-i\alpha(\mathbf{n}\cdot\mathbf{\Sigma})},$$

   where $(\Sigma_i)_{jk} = -i\epsilon_{ijk}$ are the generators of the $SO(3)$ rotations. Starting from this result, obtain the expression for the individual matrix elements of $R_{\mathbf{n}}(\alpha)$,

$$[R_{\mathbf{n}}(\alpha)]_{ij} = (\cos\alpha)\delta_{ij} + (1 - \cos\alpha)n_i n_j - (\sin\alpha)\epsilon_{ijk} n_k.$$

4. Consider a spin-$\frac{1}{2}$ particle at rest. Let the particle be in a spin-state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right).$$

What is the probability for finding the particle in the $|+\rangle$ state after a rotation by an angle α around an axis $\mathbf{n}$, when

$$\text{(i) } \mathbf{n} = \hat{\mathbf{e}}_z \qquad \text{(ii) } \mathbf{n} = \hat{\mathbf{e}}_x \qquad \text{(iii) } \mathbf{n} = \frac{1}{\sqrt{3}}\left(\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y + \hat{\mathbf{e}}_z\right)$$

What if the initial state is $|\Psi\rangle = |+\rangle$?