Problem set 8 Return before **16 on Thursday 15.3. to the box labeled FYST530**

1. Starting from the discretized version of the path integral

$$G(x'', t''; x, t) = \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\delta t}\right)^{N/2} \prod_{i=1}^{N-1} \int_{-\infty}^{\infty} \mathrm{d}x_i e^{\frac{i}{\hbar}\sum_{i=0}^{N-1}\delta t\left[\frac{m}{2}\left(\frac{x_{i+1}-x_i}{\delta t}\right)^2 - V(x_i)\right]},$$

compute the free particle propagator.

2. As suggested in the lecture notes p.237, derive the classical action $S_{\rm cl}$ for the harmonic oscillator with boundary conditions $x_{\rm cl}(t) = x$ and $x_{\rm cl}(t'') = x''$. You should obtain

$$S_{\rm cl}(x'', t''; x, t) = \frac{m\omega}{2\sin\omega(t''-t)}\left((x''^2 + x^2)\cos\omega(t''-t) - 2xx''\right).$$

3. To complete the calculation of harmonic oscillator propagator, we still need to show that for matrix

$$A_{jk}^{(N-1)} = \frac{m}{2\delta t}\left(2\delta_{jk} - \delta_{j+1,k} - \delta_{j,k+1}\right) - \frac{\delta t}{2} m\omega^2 \delta_{jk},$$

we have

$$\lim_{N\to\infty}\left(\frac{m}{2\delta t}\right)^{\frac{N-1}{2}} \sqrt{\frac{1}{\delta t \det A^{(N-1)}}} = \sqrt{\frac{\omega}{\sin\omega(t''-t)}},$$

where $t'' - t = N\delta t$.

Let us define $D^{(k)} \equiv \delta t (2\delta t/m)^k \det A^{(k)}$.

(a) Show that

$$D^{(k+1)} = \left[2 - (\delta t^2 \omega^2)\right] D^{(k)} - D^{(k-1)}.$$

(b) Now define a continuous variable $\tau = k\delta t$, and function $D(\tau) \equiv D^{(\tau/\delta t)}$. Show that $D(\tau)$ satisfies

$$\frac{d^2 D(\tau)}{d\tau^2} + \omega^2 D(\tau) = 0,$$

with boundary conditions $D(0) = 0$ and $dD(\tau)/d\tau\big|_{\tau=0} = 1$. In order to find the boundary conditions, note that from the definition of $G(x'', t''; x, t)$

$$\Psi(x'', t'') = \int_{-\infty}^{\infty} dx\, G(x'', t''; x, t)\, \Psi(x, t)$$

it follows that $G(x'', t''; x, t) \to \delta(x - x'')$ as $t'' - t \to 0$.

(c) Derive the result

$$D(\tau) = \frac{\sin(\omega\tau)}{\omega}.$$

4. The determinant $\det A^{(N)}$ can also be computed directly. Note that the matrix $A^{(N)}$ is of the band-form

$$A^{(N)} = \begin{pmatrix} 1 & a & 0 & 0 & 0 & \cdots \\ a & 1 & a & 0 & 0 & \cdots \\ 0 & a & 1 & a & 0 & \cdots \\ 0 & 0 & a & 1 & a & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(a) Verify the recurrence relation

$$|A^{(N)}| = |A^{(N-1)}| - a^2 |A^{(N-2)}|,$$

where $|A^{(N)}| \equiv \det A^{(N)}$, and solve this equation by substituting $|A^{(N)}| = C(a) f(a)^N$, where $C(a)$ and $f(a)$ are some yet unknown functions of $a$. You should get

$$|A^{(N+1)}| = \frac{1}{2^{N+1}} \left[ \left(1 + \frac{1-2a^2}{\sqrt{1-4a^2}}\right) \left(1+\sqrt{1-4a^2}\right)^N + \left(1 - \frac{1-2a^2}{\sqrt{1-4a^2}}\right) \left(1-\sqrt{1-4a^2}\right)^N \right].$$

(b) Using the result derived above, demonstrate that

$$\lim_{N\to\infty} \left(\frac{m}{2\delta t}\right)^{\frac{N-1}{2}} \sqrt{\frac{1}{\delta t \det A^{(N-1)}}} = \sqrt{\frac{\omega}{\sin \omega (t'' - t)}}.$$

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**Harmonic oscillator propagator, Algebraic method.**

*This is not an exercise problem, but a model answer will be distributed in the exercise session as there is a reference to this problem in the lecture notes (p. 238).*

In the previous exercises we derived an operator identity

$$C = e^A B e^{-A}, \qquad C = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots .$$

Iterating this, we easily see that

$$\begin{aligned} C^2 &= \left(e^A B e^{-A}\right)\left(e^A B e^{-A}\right) = e^A B^2 e^{-A} \\ &\vdots \\ C^n &= e^A B^n e^{-A}, \end{aligned}$$

so that eventually $e^C = e^A e^B e^{-A}$. In what follows we'll need these identities.

(a) The Hamiltonian for 1-dimensional harmonic oscillator is $\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega \hat{X}^2$. By direct calculation, verify that the Schrödinger picture time evolution operator can be written as

$$\exp(-\frac{i}{\hbar}\hat{H}t) = e^{i\omega t/2} \exp(-\alpha \hat{X}^2) \exp(-\beta \hat{P}^2) \exp(\frac{\omega t}{\hbar} \hat{P}\hat{X}) \exp(\beta \hat{P}^2) \exp(\alpha \hat{X}^2),$$

where $\alpha \equiv \frac{m\omega}{2\hbar}$ and $\beta \equiv \frac{-1}{4m\omega\hbar}$.

(b) Prove the relation

$$\exp(\frac{i}{\hbar}\gamma\hat{P}\hat{X})|p\rangle = e^{\gamma}|e^{\gamma}p\rangle,$$

where $|e^{\gamma}p\rangle$ is momentum eigenstate with eigenvalue $e^{\gamma}p$, i.e $\hat{P}|e^{\gamma}p\rangle = e^{\gamma}p|e^{\gamma}p\rangle$. Proceed as follows:

- Write

  $$\hat{P}\exp(\frac{i}{\hbar}\gamma\hat{P}\hat{X})|p\rangle = \exp(\frac{i}{\hbar}\gamma\hat{P}\hat{X})\left[\exp(-\frac{i}{\hbar}\gamma\hat{P}\hat{X})\hat{P}\exp(\frac{i}{\hbar}\gamma\hat{P}\hat{X})\right]|p\rangle,$$

  and show that the term in the square brackets is $[\cdots] = e^{\gamma}\hat{P}$. From this you should be able to deduce that

  $$\exp(\frac{i}{\hbar}\gamma\hat{P}\hat{X})|p\rangle = C_{\gamma}|e^{\gamma}p\rangle,$$

  where $C_{\gamma}$ is yet unknown constant.

- Show that $C_{\gamma} = e^{\gamma}$ by computing the matrix-element $\langle p'|\exp(-\frac{i}{\hbar}\gamma\hat{X}\hat{P})\exp(\frac{i}{\hbar}\gamma\hat{P}\hat{X})|p\rangle$ in two ways: By using the equation derived above, and then by using the fact that $[\hat{X}\hat{P},\hat{P}\hat{X}] = 0$.

(c) As a highlight of this problem, you have now all keys to compute the propagator for the harmonic oscillator

$$K(x',x,t-t') \equiv \langle x',t'|x,t\rangle = \langle x'|e^{-i\hat{H}(t-t')/\hbar}|x\rangle.$$

The result reads, with $\tau \equiv t-t'$:

$$K(x',x,t-t') = \sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega\tau)}}\exp\left\{\frac{im\omega}{2\hbar\sin(\omega\tau)}\left[(x'^2+x^2)\cos(\omega\tau) - 2x'x\right]\right\}.$$