Problem set 6 Return before 15 on Wednesday 29.2. to the box labeled FYST530

1. (a) To clarify the selection rules for dipole-transitions in a Hydrogen-like atom, do the exercise suggested at p.179 in the lecture notes: Starting from the recurrence relation for the associated Legendre functions $P_\ell^k(z)$ and the definition of spherical harmonics $Y_{\ell m}(\theta, \varphi)$, verify the two identities on p.180.

   (b) Using $\vec{\epsilon} = (\epsilon_x, \epsilon_y, 0)$, and starting from the matrix element $\langle \varphi_f | \vec{\epsilon} \cdot \hat{\vec{d}} | \varphi_i \rangle$ on p.174, verify in detail that the selection rules are obtained as suggested on p.181 in the lecture notes.

2. A 1-dimensional harmonic oscillator is subject to a time-dependent external electric field $E(t)$, so that the full Hamiltonian is

$$\hat{H}(t) = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}^2 - E(t)\hat{X}.$$

   Assume that the oscillator is initially in its n-th eigenstate and, using the first order time dependent perturbation theory, find the probability for transition to the k-th eigenstate for the following pulse shapes:

   (a) $E(t) = E_0 \exp(-t^2/\tau^2)$

   (b) $E(t) = E_0 (1 + t^2/\tau^2)^{-1}$

3. The Pauli matrices are defined as the matrix representation of the spin-$\frac{1}{2}$ operators:

$$\left\langle \frac{1}{2}, m | \hat{S}_i | \frac{1}{2}, m' \right\rangle \equiv \frac{\hbar}{2} (\sigma_i)_{mm'}.$$

   The Pauli matrices are frequently met in Quantum Mechanics and in Quantum Field Theories, and one should be aware of their properties or be able to derive them, if necessary. Here we'll derive some of these *without using their explicit matrix-form.*

   (a) Verify the identity $\epsilon_{ijk}\epsilon_{k\ell n} = \delta_{i\ell}\delta_{jn} - \delta_{in}\delta_{j\ell}$.

   (b) Starting from the operator equation $[\hat{S}_j, \hat{S}_k] = i\hbar\epsilon_{jk\ell}\hat{S}_\ell$, prove that

$$[\sigma_j, \sigma_k] = 2i\epsilon_{jk\ell}\sigma_\ell.$$

   (c) Show that $\mathrm{Tr}\,\sigma_i = 0$ and that $\sigma_i^2 = 1$ for $i = 1, 2, 3$.

4. Prove the following identities for Pauli matrices (still without using their explicit matrix-form):

   (a) $\boldsymbol{\sigma} \times \boldsymbol{\sigma} = 2i\boldsymbol{\sigma}$ $\quad (\boldsymbol{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z))$

   (b) Starting from the anticommutation relation $\sigma_x\sigma_y + \sigma_y\sigma_x = 0$, derived in the lecture notes, show that

$$\sigma_y\sigma_z + \sigma_z\sigma_y = 0 \quad \text{and} \quad \sigma_x\sigma_z + \sigma_z\sigma_x = 0.$$

   (c) $(\boldsymbol{\sigma} \cdot \mathbf{A})(\boldsymbol{\sigma} \cdot \mathbf{B}) = (\mathbf{A} \cdot \mathbf{B})\mathbb{I}_2 + i(\mathbf{A} \times \mathbf{B}) \cdot \boldsymbol{\sigma}$ $\quad$ ($\mathbf{A}$ and $\mathbf{B}$ are vectors)

   (d) $e^{i\mathbf{A}\cdot\boldsymbol{\sigma}} = \cos(|\mathbf{A}|) + i\frac{\mathbf{A}\cdot\boldsymbol{\sigma}}{|\mathbf{A}|}\sin(|\mathbf{A}|)$ $\quad$ ($\mathbf{A}$ is a vector)