1. (6 points)
   Compute the total cross-section in the Born approximation for scattering off a Yukawa potential
   $$V_{\text{Yukawa}}(r) = v_0 \frac{e^{-\kappa r}}{r}$$
   using the optical theorem
   $$\sigma_{\text{tot}} = \frac{4\pi}{k} \Im[f_k(\theta = 0)].$$
   You will need to compute the second term of the expansion of the scattering amplitude $f_k(\theta) = f_k^{(1)}(\Omega) + f_k^{(2)}(\Omega) + \cdots$. Using the integral form of the Green's function
   $$G_{\mathbf{k}}(\mathbf{r}) = -\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{\mathbf{p}^2 - \mathbf{k}^2 - i\epsilon},$$
   show that it becomes
   $$f_k^{(2)}(\Omega) = \frac{1}{2\pi^2} \left(\frac{2\mu v_0}{\hbar^2}\right)^2 \int \mathrm{d}^3 p \, \frac{1}{\mathbf{p}^2 - \mathbf{k}_i^2 - i\epsilon} \frac{1}{\kappa^2 + |\mathbf{k}_f - \mathbf{p}|^2} \frac{1}{\kappa^2 + |\mathbf{k}_i - \mathbf{p}|^2},$$
   where $\mathbf{k}_i$ and $\mathbf{k}_f$ are the initial and final momenta. The forward scattering amplitude is obtained from this expression by doing the remaining integral in the spherical coordinates with the help of the Residue theorem.

   Further hints: You should obtain the following result:
   $$f_k^{(2)}(0) = \frac{1}{2\pi} \left(\frac{2\mu v_0}{\hbar^2}\right)^2 \frac{1}{k} \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{p^2 - k^2 - i\epsilon} \frac{p}{\kappa^2 + (p-k)^2}$$
   which you can integrate using the residue theorem. When looking for the poles, it's convenient to write $p^2 - k^2 - i\epsilon = (p + k + i\epsilon')(p - k - i\epsilon')$, where $\epsilon' = \frac{\epsilon}{2k} \to 0$.

2. Consider the case of low-energy scattering from a spherical δ-function shell
   $$V(r) = \alpha\delta(r - a),$$
   where α and a are constants. Calculate the scattering amplitude $f(\theta)$, the differential cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$, and the total cross-section σ. Since we are looking the low-energy limit you may assume that $ka \ll 1$, so that only s-wave scattering contributes significantly. Express your result in terms of dimensionless quantity $\beta \equiv 2ma\alpha/\hbar^2$. (*Answer*: $\sigma = 4\pi a^2\beta^2/(1+\beta)^2$)

3. Consider an attractive, spherically symmetric potential given by

$$V(r) = \begin{cases} -V_0, & r < a \\ 0, & r > a \end{cases},$$

where $V_0 > 0$. Compute the s-wave phase shift $\delta_0(k)$, and draw its behaviour when V_0 is such that

(a) no bound states exist.

(b) there is only one bound state.

Sketch also the behaviour of scattering amplitude $f_\ell(k)$ in the complex plane, so called Argand-plot.

(c) Show that the scattering energy E has to fullfill the condition

$$\kappa = -k \tan(ka) \tan(\kappa a), \quad \text{where} \quad k^2 = \frac{2mE}{\hbar^2} \quad \kappa^2 = \frac{2m}{\hbar^2}(E + V_0),$$

in order to achieve a resonance in the s-wave scattering.

Extra (no points): Has something been left unclear so far in the course? Ask something about the course content or exercises, and we can try to clarify it more in the lectures or the exercise sessions.