FYST530 Quantum Mechanics II Spring 2012

Problem set 11 Return before **15 on Wednesday 11.4.** to the box labeled FYST530

1. Contributions to the magnetic moment of an atom or nucleus arise from the orbital motion of the charged particles and the intrinsic spins of all particles in the system. For the Hydrogen atom, the magnetic moment operator $\hat{\mathbf{m}}$ is, to a good approximation,

$$\hat{\mathbf{m}} = -\frac{e}{2m_e}\left(\hat{\mathbf{L}} + 2\hat{\mathbf{S}}\right),$$

where $\hat{\mathbf{S}}$ refers to the spin of the electron.

(a) Verify that $\hat{\mathbf{m}}$ is a tensor operator of rank 1 (vector operator) with respect to the total angular momentum operator of the system $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$, by explicitly showing that its spherical components satisfy

$$\begin{aligned}\left[\hat{J}_z, \hat{m}_q\right] &= q\,\hat{m}_q \\ \left[\hat{J}_\pm, \hat{m}_q\right] &= \sqrt{2 - q(q \pm 1)}\,\hat{m}_{q\pm1}.\end{aligned}$$

(b) The magnetic dipole moment of the Hydrogen atom is defined as an expectation value

$$m \equiv \langle n\,\ell\,s\,j\,(m = j)|\hat{m}_z|n\,\ell\,s\,j\,(m = j)\rangle.$$

Show that

$$m = -\frac{e\hbar j}{2m_e}\left[1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)}\right].$$

2. (a) When an atom or an atomic nucleus is in excited state $\mathcal{N}^*$, it frequently decays down to a lower state $\mathcal{N}$ by emitting a quantum of electromagnetic radiation, the photon:

$$\mathcal{N}^* \to \mathcal{N}^* + \gamma.$$

Let (J', M') and (J, M) be the angular momentum quantum numbers of $\mathcal{N}^*$ and $\mathcal{N}$ respectively, and denote their other relevant quantum numbers by α' and α. The transition probabilities are given by the absolute squares of the matrix elements

$$\langle \alpha J M|\hat{H}(\Omega, \mu)|\alpha' J' M'\rangle,$$

where $\hat{H}(\Omega, \mu)$ is a certain operator describing the transition accompanied by an emission of a photon in the direction $\Omega = (\theta, \varphi)$ with polarization μ. All we need to know about $\hat{H}(\Omega, \mu)$ is that it takes the form of a series of irreducible tensor operators $\hat{T}^\lambda_\mu$. The transitions can be classified into two types: electric and magnetic ones (distinguished by opposite parities).

Usually one is unable to measure the polarization of the photon and the M quantum number of the final state of the system, and the central quantity in the measurements is the so-called *reduced transition probability*

$$B(\lambda; J' \to J) \equiv \sum_{\mu,M} |\langle \alpha J M|\hat{T}^{(\lambda)}_\mu|\alpha' J' M'\rangle|^2.$$

In this equation $\hat{T}^{\lambda}_{\mu}$ is a tensor operator of rank λ involved either in electric or in magnetic transition. Use the Wigner-Eckart theorem to derive the result

$$B(\lambda; J' \rightarrow J) = \frac{1}{2J'+1} |\langle \alpha J || \hat{T}^{(\lambda)} || \alpha' J' \rangle|^2.$$

(b) Show also that

$$\sum_{M,M'} |\langle \alpha J M | \hat{T}^{(\lambda)}_q | \alpha' J' M' \rangle|^2 = \frac{1}{2\lambda+1} |\langle \alpha J || \hat{T}^{(\lambda)} || \alpha' J' \rangle|^2.$$

How would you interpret this quantity physically in the above situation?

3. Suppose that $\hat{T}^{(k)}$ and $\hat{U}^{(k)}$ are irreducible $SU(2)$ tensor operators of rank k. Show that the scalar product of two tensor operators (q refers to the spherical components)

$$\hat{T}^{(k)} \odot \hat{U}^{(k)} \equiv \sum_{q=-k}^{+k} (-1)^q \, \hat{T}^{(k)}_q \hat{U}^{(k)}_{-q},$$

is a scalar operator. Verify that in the case of $k = 1$ this reduces to the usual scalar product $\hat{T}^{(1)} \cdot \hat{U}^{(1)}$ of two vector operators.

4. The product of two irreducible $SU(2)$ tensor operators $\hat{T}^{(k_1)}$ and $\hat{U}^{(k_2)}$ is defined by

$$V^{(k)}_q = \left[\hat{T}^{(k_1)} \otimes \hat{U}^{(k_2)} \right]^{(k)}_q \equiv \sum_{q_1 q_2} \hat{T}^{(k_1)}_{q_1} \hat{U}^{(k_2)}_{q_2} \, {}_u\langle k_1 k_2 q_1 q_2 | k_1 k_2 k q \rangle_c.$$

(a) Show that $V^{(k)}_q$ is a tensor operator of rank k.

(b) Set $k_1 = k_2 = 1$ and form the tensor products $V^{(0)}_q$ and $V^{(1)}_q$. Show that $V^{(0)}_q$ is proportional to $\hat{T}^{(1)} \odot \hat{U}^{(1)}$, and that the components of $V^{(1)}_q$ are proportional to the spherical components of the usual cross-product $\hat{T}^{(1)} \times \hat{U}^{(1)}$ of two vector operators.