TTP3 .. Ex. 3

1. $$P_{1\,kl}^{\;ij} = \frac{1}{N_c}\,\delta_{ij}\,\delta_{kl} \qquad P_{8\,kl}^{\;ij} = 2\,t^a_{ji}\,t^a_{lk}$$

singlet — octet, ~ 1 gluon

Normalization: $$P_{1\,kl}^{\;ij}\,P_{1\,mn}^{\;lk} = \frac{1}{N_c^2}\,\delta_{ij}\,\underbrace{\delta_{kl}\,\delta_{lk}}_{\delta_{ll}=N_c}\,\delta_{mn}$$

$$= \frac{1}{N_c}\,\delta_{ij}\,\delta_{mn} = \underline{P_{1\,mn}^{\;ij}}$$

$$P_{8\,kl}^{\;ij}\,P_{8\,mn}^{\;lk} = 4\,t^a_{ji}\,\underbrace{t^a_{kl}\,t^b_{lk}}_{\mathrm{tr}\,t^at^b = \frac{1}{2}\delta^{ab}}\,t^b_{mn} = 2\,t^a_{ji}\,t^a_{mn} = \underline{P_{8\,mn}^{\;ij}}$$

$$P_{1\,kl}^{\;ij}\,P_{8\,mn}^{\;lk} = \frac{2}{N_c}\,\delta_{ij}\,\underbrace{\delta_{kl}\,t^a_{kl}}_{\mathrm{tr}\,t^a=0}\,t^a_{mn} = \underline{0}$$

Squares: $$P_{1\,kl}^{\;ij}\,P_{1\,ji}^{\;lk} = \frac{1}{N_c^2}\,\delta_{ij}\,\delta_{kl}\,\delta_{lk}\,\delta_{ji} = \frac{1}{N_c^2}\,\underbrace{\delta_{ii}}_{N_c}\,\underbrace{\delta_{kk}}_{N_c} = \underline{1}$$

$$P_{8\,kl}^{\;ij}\,P_{8\,ji}^{\;lk} = 4\,t^a_{ji}\,\underbrace{t^a_{lk}\,t^b_{kl}}_{\mathrm{tr}\,t^at^b = \frac{1}{2}\delta^{ab}}\,t^b_{ij} = 2\,\underbrace{\mathrm{tr}\,t^at^a}_{\frac{1}{2}(N_c^2-1)} = \underline{N_c^2-1}$$

Decomposition:

$$\text{Im} A \sim (t^b t^a)_{ji} (t^b t^a)_{lk} = I_1 P_1{}^{ij}_{kl} + I_8 P_8{}^{ij}_{kl} \quad \Big| \cdot P_1{}^{lk}_{ji}$$

$$I_1 \underbrace{P_1{}^{ij}_{kl} P_1{}^{lk}_{ji}}_{1} + I_8 \underbrace{P_8{}^{ij}_{kl} P_1{}^{lk}_{ji}}_{0} = P_1{}^{lk}_{ji} (t^b t^a)_{ji} (t^b t^a)_{lk}$$

$$I_1 = \frac{1}{N_c} \delta_{ij} \delta_{kl} (t^b t^a)_{ji} (t^b t^a)_{lk} = \frac{1}{N_c} tr(t^b t^a) tr(t^b t^a)$$

$$= \frac{1}{2^2 N_c} \delta^{ab} = \frac{N_c^2 - 1}{4 N_c}$$

similarly operating by $P_8{}^{lk}_{ji}$ we get

$$I_1 P_8{}^{lk}_{ji} P_1{}^{ij}_{kl} + I_8 P_8{}^{lk}_{ji} P_8{}^{ij}_{kl} = P_8{}^{lk}_{ji} (t^b t^a)_{ij} (t^b t^a)_{kl}$$

$$I_8 (N_c^2 - 1) = 2 t^c_{ij} t^c_{kl} \underbrace{(t^b t^a)_{ji}}_{t^b_{jm} t^a_{mi}} \underbrace{(t^b t^a)_{lk}}_{t^b_{ln} t^a_{nk}}$$

$$= 2 tr(t^c t^b t^a) tr(t^c t^b t^a)$$

To compute this, use the Fierz identity

$$t^a_{ij} t^a_{kl} = \frac{1}{2} \left(\delta_{il} \delta_{kj} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right)$$

or ask FeynCalc to compute the traces (see SUNT and SUNTrace functions)

$$\therefore I_8 (N_c^2 - 1) = \frac{1}{2N_c} - \frac{N_c}{2}$$

$$I_8 = -\frac{1}{2N_c}$$

Real part:

$$(t^b t^a)_{ji} [t^b, t^a]_{lk} = R_1 P_1{}^{ij}_{kl} + R_8 P_8{}^{ij}_{kl} \quad | P_1{}^{lk}_{ji}$$

$$R_1 = \frac{1}{N_c} \delta_{ij} \delta_{kl} (t^b t^a)_{ji} [t^b, t^a]_{lk}$$

$$= \frac{1}{N_c} tr(t^a t^b) \underbrace{tr(t^b t^a - t^a t^b)}_{=0} = \underline{0}$$

And operating by $P_8{}^{lk}_{ji}$

$$P_8{}^{lk}_{ji} (t^b t^a)_{ji} [t^b, t^a]_{lk} = R_8 \underbrace{P_8{}^{lk}_{ji} P_8{}^{ij}_{kl}}_{N_c^2 - 1}$$

$$(N_c^2-1) R_8 = 2 \, t^c_{ij} t^c_{kl} t^b_{jm} t^a_{mi} \, i f^{bad} t^d_{lk}$$

$$= 2 \, tr(t^c t^b t^a) \, tr(t^c t^d) \, i f^{bad}$$

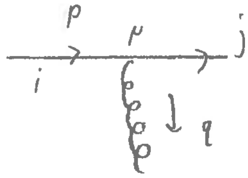
$$= i f^{bac} \underbrace{tr(t^b t^a t^c)}_{\frac{1}{4}(d^{bac} + i f^{bac})} \quad \text{Barone p. 388}$$

$$= \frac{i}{4} \underbrace{f^{bac} d^{bac}}_{=0,} - \frac{1}{4} \underbrace{f^{bac} f^{bac}}_{N_c(N_c^2-1)}$$

as antisym · sym

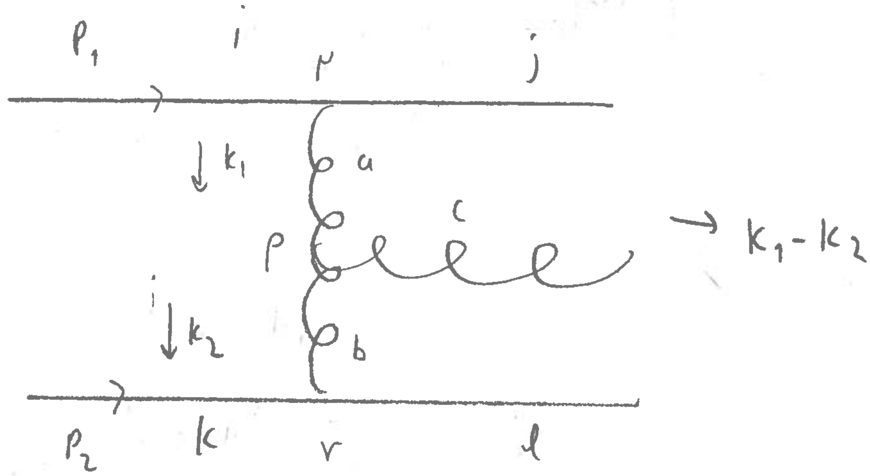
Thus $\underline{R_8 = -\frac{1}{4N_c}}$

2. Eikonal vertex:



$-2ig\, p^\mu\, t^a_{ji}$

a)



$$iA = -2ig\, P_{1,\mu}\, t^a_{ji} \quad \frac{-i}{k_1^2}(-f_{abc})$$

$$g\left[(k_1+k_2)^\rho g^{\mu\nu} + (k_1-2k_2)^\mu g^{\nu\rho} + (k_2-2k_1)^\nu g^{\mu\rho} \right]$$

$$\frac{-i}{k_2^2}\left(-2ig\, P_{2,\nu}\right) t^b_{lk}$$

use:

$$P_{1,\mu}(k_1-2k_2)^\mu = P_1^+(k_1-2k_2)^- \approx -2P_1^+k_2^-$$

$$P_{2,\nu}(k_2-2k_1)^\nu = P_2^-(k_2-2k_1)^+ \approx -2P_2^-k_1^+$$

$$\Rightarrow \quad iA = -\frac{4g^3}{k_1^2k_2^2} f_{abc}\, P_{1,\mu}P_{2,\nu}\left[g^{\mu\nu}(k_1+k_2)^\rho - 2k_2^\mu g^{\nu\rho} - 2k_1^\nu g^{\mu\rho} \right]$$

Finally write $\quad k_{1,2} = (\alpha_{1,2} P_1^+, \beta_{1,2} P_2^-, \underset{\sim}{k}_{1,2})$

$$P_{1,\mu} k_2^\mu = P_1^+ k_2^- = \beta_2 P_1^+ P_2^- = \beta_2 P_{1\mu} P_{2\nu} g^{\mu\nu}$$

$$P_{2,\nu} k_1^\nu = P_2^- k_1^+ = \alpha_1 P_2^- P_1^+ = \alpha_1 g^{\mu\nu} P_{1,\mu} P_{2,\nu}$$

And

$$k_i^2 = \alpha_i \beta_i P_1^+ P_2^- - \underset{\sim}{k}_i^2$$

For an on-shell gluon

$$0 = (k_1 - k_2)^2 = 2 \underbrace{(\alpha_1 - \alpha_2)}_{\simeq \alpha_1} \underbrace{(\beta_1 - \beta_2)}_{\simeq \beta_2} P_1^+ P_2^- - (\underset{\sim}{k}_1 - \underset{\sim}{k}_2)^2$$

$$2 \alpha_1 \beta_2 P_1^+ P_2^- \simeq (\underset{\sim}{k}_1 - \underset{\sim}{k}_2)^2$$

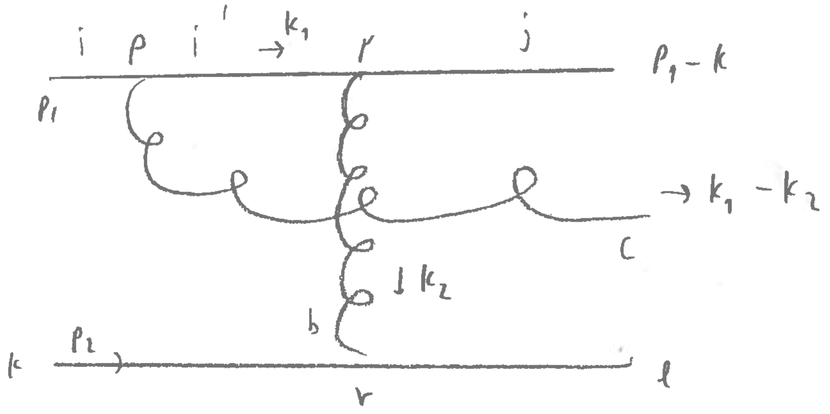
And thus $\alpha_i \beta_i P_1^+ P_2^- << (\underset{\sim}{k}_1 - \underset{\sim}{k}_2)^2$

$\therefore k_i^2 \simeq - \underset{\sim}{k}_i^2$, and

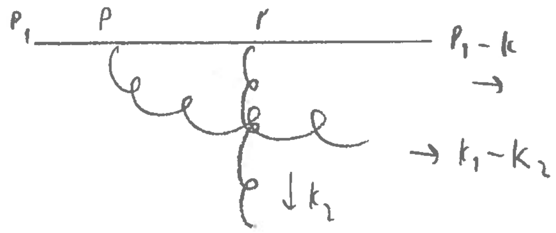
$$iA_a = - \frac{4 g^3 f_{abc} t^a_{ji} t^a_{lk}}{\underset{\sim}{k}_1^2 \underset{\sim}{k}_2^2} P_{1,\mu} P_{2,\nu} g^{\mu\nu}$$

$$\times \left[(k_1 + k_2)^\rho - 2\beta_2 P_2^\rho - 2\alpha_1 P_1^\rho \right]$$

b)



Need double eikonal vertex from lectures

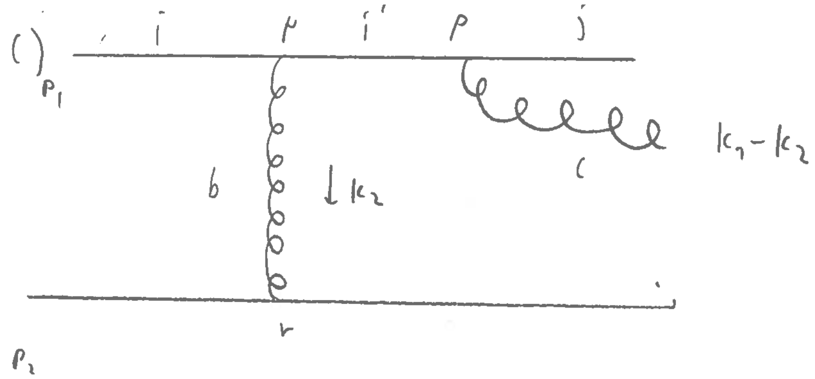


$$= \frac{i g^2}{\beta_2 s} (-2 i p_1^\mu)(-2 i p_1^\rho)$$

$$\Rightarrow \quad i A_b = \frac{i g^2}{\beta_2 s} (-2 i p_1^\mu)(-2 i p_1^\rho) \; t^b_{j i'} \, t^c_{i' i} \; \frac{-i \, g_{\mu\nu}}{k_2^2}$$

$$(-2 i g p_2^\nu) \; t^b_{l k}$$

$$= - \frac{8 i g^3}{\beta_2 s \, \underset{\sim}{k}_2^2} \; p_1^\mu p_2^\nu p_1^\rho \; g_{\mu\nu} \, [t^b t^c]_{ji} \; t^b_{lk}$$



Same double-eikonal vertex with different momenta, e.g. fermion propagator momentum is $p_1 - k_2$, but $(p_1-k)^2 \simeq (p_1 - k_1 + k_2)^2$

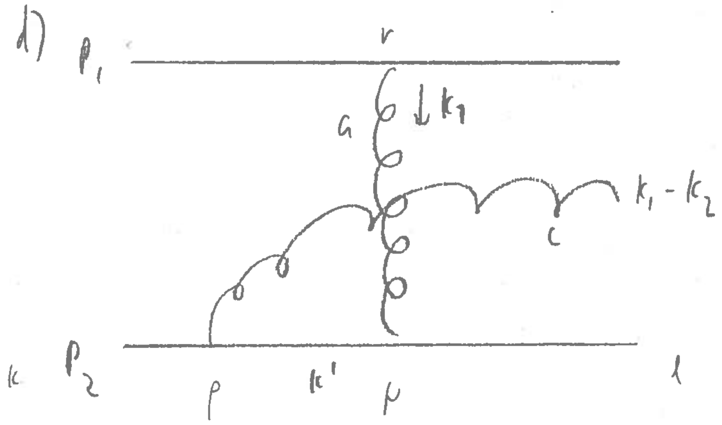
$\Rightarrow$ only difference to diagram b is the color structure which is now

$$t^c_{ji'} \, t^b_{i'i} \, t^b_{lk}$$

Thus

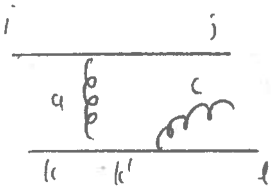
$$iA_b + iA_c = -\frac{8ig^3}{\beta_2 s \underset{\sim}{k}_2^2} \, p_1^\mu p_2^\nu p_1^\rho g_{\mu\nu} \underbrace{\left([t^b t^c]_{ji} + [t^c t^b]_{ji} \right) t^b_{lk}}_{[t^b, t^c]_{ji} \, t^b_{lk}}$$

d)



same eikonal vertex as in diagram b,
but now $p_1 \to p_2$, $k_2 \to -k_1$, and different colors.

Similarly in diagram e) =



differs from c) by $p_1 \to p_2$, $k_2 \to -k_1$ and color structure.

Thus we directly get

$$iA_d + iA_e = -\frac{g\, p_2^{\rho}\, p_1^{\nu}\, p_2^{\nu}}{-\alpha_1 s\, \underline{k}_1^2}\, g_{\mu\nu} \underbrace{\left(t^a_{lk'} t^c_{k'k} t^a_{ji} + t^c_{lk'} t^a_{k'k} t^a_{ji} \right)}_{[t^a, t^c]_{lk} t^a_{ji}}$$

$\beta_2 \to -\alpha_1$

Now

$$i(A_b + A_c + A_d + A_e) = -\frac{8ig^3}{s\,\underset{\sim}{k}_1^2\,\underset{\sim}{k}_2^2}\, p_1^\mu p_2^\nu g_{\mu\nu}$$

$$\Big[\frac{\underset{\sim}{k}_1^2}{\beta_2}\underbrace{[t^b, t^c]_{ji}\, t^b_{lk}}_{i f^{abc} t^a_{ji} t^b_{lk}}\, p_1^\rho - \frac{\underset{\sim}{k}_2^2}{\alpha_1}\underbrace{[t^a, t^c]_{lk}\, t^a_{ji}}_{i \underbrace{f^{acb}}_{-i f^{abc}} t^b_{lk} t^a_{ji}} p_2^\rho\Big]$$

$$= -\frac{8ig^3}{s\,\underset{\sim}{k}_1^2\underset{\sim}{k}_2^2}\, g_{\mu\nu}\, p_1^\mu p_2^\nu \left[\frac{\underset{\sim}{k}_1^2}{\beta_2}p_1^\rho + \frac{\underset{\sim}{k}_2^2}{\alpha_1}p_2^\rho\right] i f^{abc} t^a_{ji} t^b_{lk}$$

And

$$i(A_a + A_b + A_c + A_d + A_e) = -\frac{\overbrace{4g^3\, p_{1,\mu}\, p_{2,\nu}}^{\text{eikonal vertex + propagators}}}{\underset{\sim}{k}_1^2\,\underset{\sim}{k}_2^2}\, g_{\mu\nu}\Big[(k_1+k_2)^\rho - 2\beta_2 p_2^\rho - 2\alpha_1 p_1^\rho$$

$$- \frac{2\underset{\sim}{k}_1^2}{\beta_2 s}p_1^\rho - \frac{2\underset{\sim}{k}_2^2}{\alpha_1 s}p_2^\rho\Big]\ \underbrace{f^{abc}\, t^a_{ji}}_{\text{color factors}}\, t^b_{lk}$$

Thus we can read c^ρ,

$$c^\rho = (k_1+k_2)^\rho - 2\beta_2 p_2^\rho - 2\alpha_1 p_1^\rho$$

$$- \frac{2\underset{\sim}{k_1}^2}{\beta_2 s} p_1^\rho - \frac{2\underset{\sim}{k_2}^2}{\alpha_1 s} p_2^\rho \Big]$$

In MRK

$$c^+ = -\Big(k_1^+ + \underbrace{k_2^+}_{\approx 0} - \underbrace{2\alpha_1 p_1^+}_{2k_1^+} - \frac{2\underset{\sim}{k_1}^2}{\beta_2 s} p_1^+ \Big)$$

$\nwarrow$ $s = 2p_1^+ p_2^-$

$\beta s = 2 p_1^+ k_2^-$

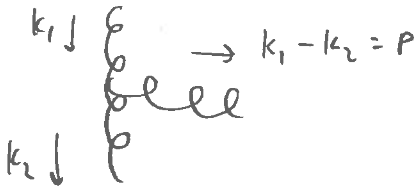
$$= k_1^+ + \frac{\underset{\sim}{k_1}^2}{k_2^-}$$

$$c^- = -\Big(k_1^- + k_2^- - 2\overbrace{\beta_2 p_2^-}^{k_2^-} - \frac{2\underset{\sim}{k_2}^2}{\alpha_1 s} p_2^- \Big)$$

$$= k_2^- + \frac{\underset{\sim}{k_2}^2}{k_1^+}$$

$$\underset{\sim}{c} = -(k_1+k_2)_T$$

3. $C^{\rho} = (C^{+}, C^{-}, C_{T}) = \left(k_1^{+} + \dfrac{\underset{\sim}{k}_1^{2}}{k_2^{-}},\ k_2^{-} + \dfrac{\underset{\sim}{k}_2^{2}}{k_1^{+}},\ -(\underset{\sim}{k}_1 + \underset{\sim}{k}_2) \right)$



MRK: $k_1^{+} \gg k_2^{+}$

$k_2^{-} \gg k_1^{-}$

a) $C^{+} = p^{+} + \dfrac{\underset{\sim}{k}_1^{2}}{-p^{-}} = p^{+} - \dfrac{\underset{\sim}{k}^{2}}{p^{-}}$

$C^{-} = -p^{-} + \dfrac{\underset{\sim}{k}_2^{2}}{p^{+}}$ $\qquad C_{T} = -(\underset{\sim}{k}_1 + \underset{\sim}{k}_2)$

b) Recall: $a \cdot b = a^{+} b^{-} + a^{-} b^{+} - \underset{\sim}{a} \cdot \underset{\sim}{b}$

$p^{\rho} C_{\rho}(k_1, k_2) = p^{+} C^{-} + C^{-} p^{+} - \underset{\sim}{p} \cdot \underset{\sim}{C}$

$= p^{+}\left(-p^{-} + \dfrac{\underset{\sim}{k}_2^{2}}{p^{+}}\right) + p^{-}\left(p^{+} + \dfrac{\underset{\sim}{k}_1^{2}}{-p^{-}}\right) - \underset{\sim}{p} \cdot (-\underset{\sim}{k}_1 - \underset{\sim}{k}_2)$

$= \underset{\sim}{k}_2^{2} - \underset{\sim}{k}_1^{2} - (\underset{\sim}{k}_1 - \underset{\sim}{k}_2) \cdot (-\underset{\sim}{k}_1 - \underset{\sim}{k}_2)$

$= \underset{\sim}{k}_2^{2} - \underset{\sim}{k}_1^{2} + \underset{\sim}{k}_1^{2} - \underset{\sim}{k}_2^{2} = 0$

2. c)

$$C^{\rho}(k_1,k_2)\, C_{\rho}(k_1,k_2) = 2\, C^+ C^- - C_T \cdot C_T$$

$$= 2\left(p^+ - \frac{\tilde{k}_1^2}{p^-}\right)\left(-p^- + \frac{\tilde{k}_2^2}{p^+}\right) - (-\tilde{k}_1 - \tilde{k}_2)^2$$

$$= 2\left[-p^+p^- + \tilde{k}_2^2 + \tilde{k}_1^2 - \frac{\tilde{k}_1^2\, \tilde{k}_2^2}{p^+p^-}\right] - (-\tilde{k}_1 - \tilde{k}_2)^2$$

on-shell gluon so

$$2p^+p^- - \tilde{p}^2 = 0$$

$$p^+p^- = \frac{1}{2}\tilde{p}^2$$

thus

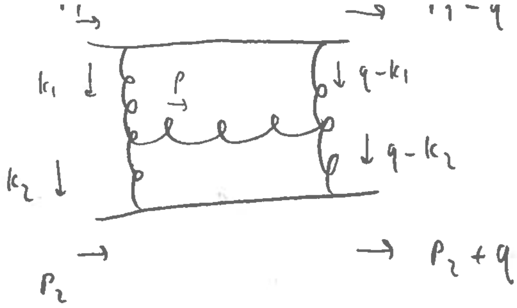
$$C^2 = -\tilde{p}^2 + 2\tilde{k}_1^2 + 2\tilde{k}_2^2 - 4\,\frac{\tilde{k}_1^2\, \tilde{k}_2^2}{\tilde{p}^2} - (\tilde{k}_1 + \tilde{k}_2)^2$$

$$= -(\tilde{k}_1^2 + \tilde{k}_2^2 - 2\tilde{k}_1\cdot\tilde{k}_2) + 2\tilde{k}_1^2 + 2\tilde{k}_2^2 - (\tilde{k}_1^2 + \tilde{k}_2^2 - 2\tilde{k}_1\cdot\tilde{k}_2)$$

$$- 4\,\frac{\tilde{k}_1^2\, \tilde{k}_2^2}{\tilde{p}^2}$$

$$= -4\,\frac{\tilde{k}_1^2\, \tilde{k}_2^2}{\tilde{p}^2}$$

3d



MRK: $q^+ \ll p_1^+$

$q^- \ll p_2^-$

Here now we have to replace $p^+ \to -p^+$, $p^- \to -p^-$,

$$C^p(k_1,k_2)\,\overbrace{C_p(q-k_1,\,q-k_2)}$$

$$= \left(p^+ - \frac{\mathbf{k}_1^2}{p^-}\right)\left(+p^- + \frac{(\mathbf{q}-\mathbf{k}_1)^2}{-p^+}\right) + \left(-p^- + \frac{\mathbf{k}_2^2}{p^+}\right)\left(-p^+ - \frac{(\mathbf{q}-\mathbf{k}_1)^2}{-p^-}\right)$$

$$- \left(-\mathbf{k}_1 - \mathbf{k}_2\right)\cdot\left(-\mathbf{q} + \mathbf{k}_1 - \mathbf{q} + \mathbf{k}_2\right)$$

$$= \underbrace{2p^+p^-}_{(\mathbf{k}_1-\mathbf{k}_2)^2} - (\mathbf{q}-\mathbf{k}_2)^2 - \mathbf{k}_1^2 + \frac{\mathbf{k}_1^2(\mathbf{q}-\mathbf{k}_2)^2}{p^+p^-} - (\mathbf{q}-\mathbf{k}_1)^2 - \mathbf{k}_2^2 + \frac{\mathbf{k}_2^2(\mathbf{q}-\mathbf{k}_1)^2}{p^+p^-}$$

$$+ (\mathbf{k}_1+\mathbf{k}_2)^2 + (\mathbf{k}_1+\mathbf{k}_2)\cdot(-2\mathbf{q})$$

$$= \frac{4\mathbf{k}_1^2\mathbf{k}_2^2 - 4\mathbf{k}_1^2\,\mathbf{k}_2\cdot\mathbf{q} - 4\mathbf{k}_2^2\,\mathbf{k}_1\cdot\mathbf{q} + 4\,\mathbf{k}_1\cdot\mathbf{k}_2\,\mathbf{q}^2}{(\mathbf{k}_1-\mathbf{k}_2)^2}$$

$$= \frac{2}{(\mathbf{k}_1-\mathbf{k}_2)^2}\left[-\mathbf{k}_1^2(\mathbf{k}_2-\mathbf{q})^2 + \mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2 - \mathbf{k}_1^2\mathbf{q}^2 + \mathbf{k}_2^2\mathbf{q}^2 + 2\mathbf{k}_1\cdot\mathbf{k}\,\mathbf{q}\right]$$

$$= \frac{-2}{(\mathbf{k}_1-\mathbf{k}_2)^2}\left[-\mathbf{k}_1^2(\mathbf{k}_2-\mathbf{q})^2 - \mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2 + \mathbf{q}^2(\mathbf{k}_1-\mathbf{k}_2)^2\right]$$

$$= -2\left[\underset{\sim}{q}^2 - \frac{\underset{\sim}{k}_1^2 (\underset{\sim}{q} - \underset{\sim}{k}_2)^2}{(\underset{\sim}{k}_1 - \underset{\sim}{k}_2)^2} - \frac{\underset{\sim}{k}_2^2 (\underset{\sim}{q} - \underset{\sim}{k}_1)^2}{(\underset{\sim}{k}_1 - \underset{\sim}{k}_2)^2} \right]$$

Thus
$$\frac{C^{\rho}(k_1, k_2)\; C_{\rho}(q - k_1,\; q - k_2)}{\underset{\sim}{k}_1^2\, \underset{\sim}{k}_2^2\, (\underset{\sim}{q} - \underset{\sim}{k}_1)^2 (\underset{\sim}{q} - \underset{\sim}{k}_2)^2}$$

$$= -2\left[\frac{\underset{\sim}{q}^2}{\underset{\sim}{k}_1^2\, \underset{\sim}{k}_2^2\, (\underset{\sim}{q} - \underset{\sim}{k}_1)^2 (\underset{\sim}{q} - \underset{\sim}{k}_2)^2} - \frac{1}{\underset{\sim}{k}_2^2\, (\underset{\sim}{q} - \underset{\sim}{k}_1)^2 (\underset{\sim}{k}_1 - \underset{\sim}{k}_2)^2} \right.$$

$$\left. - \frac{1}{\underset{\sim}{k}_1^2\, (\underset{\sim}{q} - \underset{\sim}{k}_2)^2 (\underset{\sim}{k}_1 - \underset{\sim}{k}_2)^2} \right]$$

4. $f(w) = \int_0^\infty dY \, e^{-wY} \int_0^{Y} dY_1 \int_0^{Y_1} dY_2 \cdots \int_0^{Y_n} dY_n \; e^{(Y-Y_1)\varepsilon_1} \cdots e^{(Y_{n-1}-Y_n)\varepsilon_n} e^{Y_n \varepsilon_{n+1}}$

where $\varepsilon_i = \varepsilon(k_i) + \varepsilon(q-k_i)$

write $Y_{i-1} - Y_i = z_i$ $\quad (z_1 = Y - Y_1)$

Notice: $\sum_i z_i = Y_0 - Y_n \Rightarrow Y_0 = \sum_i z_i + Y_n$

$$f(w) = \int_0^\infty dz_1 \cdots dz_n \; e^{z_1(\varepsilon_1 - w)} e^{z_2(\varepsilon_2 - w)} \cdots e^{z_n(\varepsilon_n - w)}$$

$$\int_0^\infty dY_n \, e^{Y_n(\varepsilon_{n+1} - w)} \qquad | \text{ factorizes!}$$

$$= \prod_{i=1}^{n+1} \frac{1}{w - \varepsilon_i}$$