FYSH555 EX. 2

1. $[\nabla^2 - U(\bar{r}) + k^2]\,\psi(\bar{r}) = 0$ (Schroedinger eq. with $k^2 = \frac{2mE}{\hbar^2}$, $U = \frac{2m}{\hbar^2} V$

a) $$\psi(r) = e^{ik\cdot r} - \frac{1}{4\pi}\int d^3r' \, \frac{e^{ik|r-r'|}}{|r-r'|} U(r')\psi(r')$$

when $|\bar{r}|$ is large, integral oscillates to 0 and $\psi \to e^{ik\cdot r}$

$$[\nabla^2 - U(r) + k^2]\, e^{ik\cdot r} = [-k^2 - U(r) + k^2]\, e^{ikr} = -U(r)e^{ik\cdot r}$$

Also $$\nabla^2 \frac{e^{ik|r-r'|}}{|r-r'|} = -k^2 \frac{e^{ik|r-r'|}}{|r-r'|}$$ for $r \neq r'$, ok here as we are interested in limit $r \to \infty$

Brute force explanation: write $|r-r'| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta}$,

$$\nabla^2 = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + \text{azimuth}$$

and simplify.

Or use the chain rule.

Thus

$$[\nabla^2 - U(r) + k^2]\,\psi(r) = -U(r)e^{ikr} + U(r)\int d^3r' \frac{e^{ik|r-r'|}}{|r-r'|} U(r')\psi(r')$$

$$= -U(r)\,\psi(r) \to 0 \quad \text{at large } r.$$

as $|\psi(r)|$ is bounded.

1 a) A better solution

$$\nabla^2 \frac{e^{ik|\bar r-\bar r'|}}{|\bar r-\bar r'|} = \frac{1}{|\bar r-\bar r'|}\nabla^2 e^{ik|\bar r-\bar r'|}$$

$$+ e^{ik|r-r'|}\underbrace{\nabla^2\frac{1}{|\bar r-\bar r'|}}_{-4\pi\delta^3(\bar r-\bar r')}$$

$$+ 2\left(\nabla e^{ik|\bar r-\bar r'|}\right)\cdot\left(\nabla\frac{1}{|\bar r-\bar r'|}\right)$$

| write $\nabla$ in spherical coordinates

$$= \frac{1}{|\bar r-\bar r'|}\left(-k^2 e^{ik|\bar r-\bar r'|} + 2ik\frac{e^{ik|r-r'|}}{|r-r'|}\right)$$

$$-4\pi e^{ik|\bar r-\bar r'|}\delta(\bar r-\bar r') - 2ik\frac{e^{ik|\bar r-\bar r'|}}{|\bar r-\bar r'|^2}$$

Thus

$$[\nabla^2 - U(\bar r)+k^2]\psi(\bar r) = (-k^2-U(\bar r)+k^2)e^{ik\cdot r}$$

$$-\frac{1}{4\pi}\int d^3\bar r' U(\bar r')\left(-k^2\frac{e^{ik|\bar r-\bar r'|}}{|\bar r-\bar r'|} - 4\pi e^{ik|r-r'|}\delta(r-r')\right)\psi(r')$$

$$+\frac{1}{4\pi}(-U(\bar r)+k^2)\int d^3\bar r'\frac{e^{ik|\bar r-\bar r'|}}{|\bar r-\bar r'|}U(r')\psi(r')$$

$$= (-k^2 + U(\bar r) - U(\bar r) + k^2)\psi(\bar r) = 0$$

b) Large $|\bar{r}|$, write

$$\psi(r) = e^{i\bar{k}\cdot\bar{r}} + f(k,k')\,\frac{e^{ikr}}{r}$$

initial momentum: $\bar{k}$

final —"— : $\bar{k}'$

At large distances $r = |\bar{r}|$

$$\frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} \simeq \frac{1}{r}\, e^{ik|\bar{r}-\bar{r}'|},$$

and $|\bar{r}-\bar{r}'| = \sqrt{\bar{r}^2 + \bar{r}'^2 - 2\bar{r}\cdot\bar{r}'} = r\sqrt{1 + \underbrace{\frac{r'^2}{r^2}}_{\to 0} - 2\frac{\bar{r}}{r^2}\cdot r'}$

$$\approx r - \hat{r}\cdot\bar{r}' \qquad \hat{r} = \frac{\bar{r}}{r},$$

and $k|r-r'| \simeq kr - \underbrace{k\hat{r}}_{\bar{k}'}\cdot\bar{r}' = kr - k'\cdot r'$

Thus

$$f(k,k') \simeq -\frac{1}{4\pi}\int d^3r'\, e^{-i\bar{k}'\cdot\bar{r}'}\, U(\bar{r}')\,\psi(\bar{r}')$$

Substitute $\psi(r) \simeq \exp\left[i\bar{k}\cdot\bar{r} - \frac{i}{2k}\int dz'\, U(x,y,z')\right]$

and assume $\bar{q}_z = (\bar{k}'-\bar{k})_z = 0$

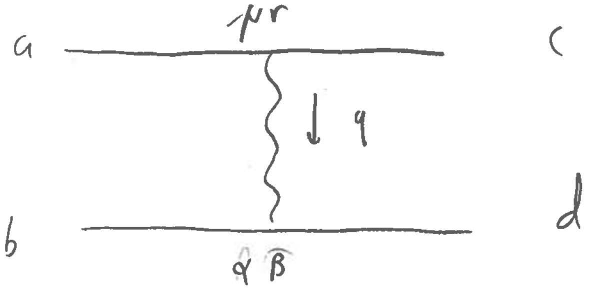
$$f(\bar{k},\bar{k}') = -\frac{1}{4\pi}\int d^3r' \, e^{-i\bar{k}'\cdot\bar{r}} \, e^{i\bar{k}\cdot r'} \underbrace{v(\bar{r}')\, e^{-\frac{i}{2k}\int_{-\infty}^{z} dz'\, v(\bar{r}_T', z')}}_{\frac{-2k}{i}\,\partial_z\, e^{-\frac{i}{2k}\int_{-\infty}^{z} dz'\, v(\bar{r}_T', z')}}$$

$$= \frac{k}{2\pi i}\int d^2r'_T \, e^{i\bar{q}_T\cdot\bar{r}_T'} \underbrace{\int_{-\infty}^{\infty} dz\, \partial_z\, e^{-\frac{i}{2k}\int_{-\infty}^{z} dz'\, v(\bar{r}_T', z')}}_{\underbrace{e^{-\frac{i}{2k}\int_{-\infty}^{\infty} dz'\, v(\bar{r}_T', z')}}_{e^{i\chi(r_T')}} - 1}$$

$$= \frac{ik}{2\pi}\int d^2r_T' \, e^{i\bar{q}_T\cdot\bar{r}_T'} \underbrace{\left[1 - e^{i\chi(r_T')}\right]}_{\Gamma(r_T)}$$

$$= \frac{ik}{2\pi}\int d^2r_T' \, e^{i\bar{q}_T\cdot r_T'}\, \Gamma(r_T)$$ (2d Fourier transfer)

2. Consider a graviton exchange




Propagator: $\dfrac{i}{2q^2}\left[g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha}-g^{\mu\nu}g^{\alpha\beta}\right]$

Vertex : $\dfrac{-i\kappa}{2}\left[p_\mu p'_\nu + p'_\mu p_\nu - g_{\mu\nu}\, p\cdot p'\right]$ (momenta inside the vertex)

Thus, for scalar non-identical particle scattering

$$-iA = \left[p_{a\mu}(-p_c)_\nu + (-p_c)_\mu p_{a\nu} - g_{\mu\nu}\, p_a\cdot(-p_c)\right]$$

$$\frac{i}{2q^2}\left[g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha}-g^{\mu\nu}g^{\alpha\beta}\right]$$

$$\left[(p_b)_\alpha(-p_d)_\beta + (-p_d)_\alpha p_{b\beta} - g_{\alpha\beta}\, p_b\cdot(-p_c)\right]$$

Using FeynCalc we get (see attached notebook)

$$-iA = \kappa^2 i\,\frac{s^2+u^2-t^2}{2t}, \quad \text{and} \quad |A|^2 = \kappa^4\frac{s^2u^2}{t^2}$$

using $s+t+u=0$

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}|A|^2 = \frac{\kappa^4}{16\pi}\,\frac{u^2}{t^2} \longrightarrow \frac{\kappa^4}{16\pi}\,\frac{s^2}{t^2} \rightarrow \infty$$

at large $s$.

As $\left[\frac{d\sigma}{dt}\right] = \frac{GeV^{-2}}{GeV^{2}} = GeV^{-4}$,

we have

$$[|a|^4] = GeV^{-4} \Rightarrow [|a|] = GeV^{-1}$$

($\Leftrightarrow$ $A$ is dimensionless, ok! )

In[1]:= `Quiet[<< HighEnergyPhysics`FeynCalc`]`

Loading FeynCalc from /home/hejajama/.Mathematica/Applications/HighEnergyPhysics
**FeynCalc** 8.2.0 For help, type ?FeynCalc, open `FeynCalcRef8.nb` or visit www.feyncalc.org
Loading FeynArts, see www.feynarts.de for documentation
FeynArts 3.7 patched for use with FeynCalc

In[2]:=

In[3]:=
```
SetMandelstam[s, t, u, Pa, Pb, -Pc, -Pd, 0, 0, 0, 0];
```

In[4]:=
```
propagator =
  I / (2 ScalarProduct[Pa - Pc, Pa - Pc]) * (MetricTensor[μ, α] MetricTensor[ν, β] +
     MetricTensor[μ, β] MetricTensor[ν, α] - MetricTensor[μ, ν] MetricTensor[α, β]);
```

In[5]:=
```
amplitude =
 κ * (FourVector[Pa, μ] FourVector[-Pc, ν] + FourVector[-Pc, μ] FourVector[Pa, ν] -
    MetricTensor[μ, ν] * ScalarProduct[Pa, -Pc]) * propagator * κ *
  (FourVector[Pb, α] FourVector[-Pd, β] + FourVector[-Pd, α] FourVector[Pb, β] -
    MetricTensor[α, β] * ScalarProduct[Pa, -Pc])
```

Out[5]=
$$\frac{i\,\kappa^2\left(g^{\alpha\nu}g^{\beta\mu}+g^{\alpha\mu}g^{\beta\nu}-g^{\alpha\beta}g^{\mu\nu}\right)\left(-\frac{1}{2}\,t\,g^{\mu\nu}-\mathrm{Pa}^{\nu}\,\mathrm{Pc}^{\mu}-\mathrm{Pa}^{\mu}\,\mathrm{Pc}^{\nu}\right)\left(-\frac{1}{2}\,t\,g^{\alpha\beta}-\mathrm{Pb}^{\beta}\,\mathrm{Pd}^{\alpha}-\mathrm{Pb}^{\alpha}\,\mathrm{Pd}^{\beta}\right)}{2\,(\mathrm{Pa}-\mathrm{Pc})^2}$$

In[6]:=
```
contracted = amplitude // Contract // DiracSimplify // FullSimplify
```

Out[6]=
$$\frac{i\,\kappa^2\left(s^2-t^2+u^2\right)}{2\,t}$$

In[7]:=
```
TrickMandelstam[contracted * ComplexConjugate[contracted], {s, t, u, 0}]
```

Out[7]=
$$\frac{\kappa^4\,s^2\,u^2}{t^2}$$

3. $\operatorname{Im} A(s,t) = s \quad \text{for} \quad s > s_+$

$$\operatorname{Re} A = \frac{s^2}{\pi} P \int_{s_+}^{\infty} ds' \frac{\operatorname{Im} A(s',t)}{s'^2(s'-s)}$$

$$= \frac{s^2}{\pi} P \int_{s_+}^{\infty} ds' \underbrace{\frac{1}{s'(s'-s)}}_{\frac{1}{s}\left(\frac{-1}{s'} + \frac{1}{s'-s}\right)}$$

$$= \frac{s}{\pi} \left[ \underbrace{P \int_{s_+}^{\infty} ds' \frac{-1}{s'}}_{\substack{\wedge \, 0 \\ \text{divergences}} \quad -\lim_{b\to\infty} \left. \ln s' \right|_{s_+}^{b}} + \underbrace{P \int_{s_+}^{\infty} ds' \frac{1}{s'-s}}_{\int_{s_+}^{s-\varepsilon} ds' \frac{1}{s'-s} + \int_{s+\varepsilon}^{\infty} ds' \frac{1}{s'-s}} \right]$$

$$= \frac{s}{\pi} \lim_{\substack{b\to\infty \\ \varepsilon\to 0}} \left[ -\ln b + \ln s_+ + \underbrace{\ln(s-\varepsilon-s)}_{\ln \varepsilon - i\pi} - \ln(s_+-s) + \ln(b-s) - \underbrace{\ln(s+\varepsilon-s)}_{\ln \varepsilon} \right]$$

$$= \frac{s}{\pi} \lim_{b\to\infty} \left[ -\ln b + \ln(b-s) - \underbrace{\ln(s_+-s)}_{\ln(s-s_+) - i\pi} + \ln s_+ - i\pi \right]$$

$$= \frac{s}{\pi} \lim_{b\to\infty} \left[ \ln \frac{b-s}{b} + \ln \frac{s_+}{s-s_+} \right] = \frac{s}{\pi} \ln \frac{s_+}{s-s_+}$$

Thus

$$A = is + \frac{s}{\pi} \ln \frac{s_+}{s - s_+} = is - \frac{s}{\pi} \ln \overbrace{\left( \frac{s}{s_+} - 1 \right)}^{>0}$$

High-energy limit

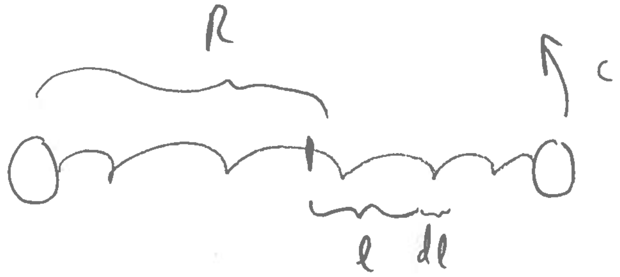
$$A = is + \frac{s}{\pi} \ln \frac{1}{s} = -\frac{s}{\pi} \ln(-s)$$

Now

$$\mathrm{Im}\, \frac{-s}{\pi} \ln(-s) = \mathrm{Im}\left[ -\frac{s}{\pi} \left( \ln s - \pi i \right) \right] = s$$

ok!

4.




$$v(l) = \frac{l}{R} \quad , \text{set } c=1$$

$$\frac{1}{2}E_{lab} = \int_0^R \frac{\kappa\, dl}{\sqrt{1-v^2}} = \int_0^R \frac{\kappa\, dl}{\sqrt{1-l^2/R^2}} = \frac{\kappa \pi R}{2}$$

$$E_{lab} = \pi \kappa R = m$$

Angular momentum:

Recall that $v = \frac{p}{E}$, thus

$$p = vE = \frac{l}{R}E \quad , \text{ and}$$

$$dp = \frac{l}{R}dE = \frac{l}{R}\frac{\kappa}{\sqrt{1-l^2/R^2}}dl, \quad \text{and we get}$$

$$\frac{1}{2}J = \int_0^R l\,dp = \frac{\kappa}{R}\int_0^R \frac{l^2}{\sqrt{1-l^2/R^2}} = \frac{\pi \kappa R^2}{4}$$

$$J = \frac{1}{2}\pi\kappa R^2 = \frac{1}{2}\frac{m^2}{\pi\kappa} \quad , \text{ thus } J \sim m^2$$

$$\alpha' = \frac{1}{2\pi\kappa}$$

5. Veneziano amplitude

$$A(s,t) = \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \qquad \alpha(x) = \alpha_0 + \alpha' x$$

$$= \frac{\Gamma(-\alpha_0 - \alpha' s)\,\Gamma(-\alpha_0 - \alpha' t)}{\Gamma(-2\alpha_0 - \alpha'(s+t))}$$

a)

As $\Gamma(x) \neq 0$ for all $x \in \mathbb{R}$, the poles of the amplitude are poles of $\Gamma(x)$, that is, $x = 0, -1, -2, \ldots$

$n \in \mathbb{N}$

Thus $-\alpha_0 - \alpha' s = -n$ or $-\alpha_0 - \alpha' t = -n$

$$s = \frac{n - \alpha_0}{\alpha'}$$

$\Rightarrow$ poles for particles with masses $m^2 = \dfrac{n - \alpha_0}{\alpha'}$, if $n > \alpha_0$

(s-channel resonance scattering)

$$t = \frac{n - \alpha_0}{\alpha'} \qquad \text{when} \qquad n < \alpha_0$$

Due to the crossing symmetry there must also be poles.

b) $\Gamma(n) = (n-1)!$ , $n! \approx n^n e^{-n}$

$$\frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \approx \Gamma(-\alpha(t))\,\frac{(-\alpha(s)-1)!}{(-\alpha(s)-\alpha(t)-1)!}$$

$$\approx \Gamma(-\alpha(t))\,\frac{(-\alpha(s)-1)^{-\alpha(s)-1}\,\cancel{e^{\alpha(s)+1}}}{(-\alpha(s)-\alpha(t)-1)^{-\alpha(s)-\alpha(t)-1}\,e^{\alpha(s)+\alpha(t)\cancel{+1}}}$$

$$\approx \Gamma(-\alpha(t))\,e^{-\alpha(t)}\,\frac{(-\alpha(s)-1)^{-\alpha(s)-1}}{(\underbrace{-\alpha(s)-\alpha(t)-1}_{\approx -\alpha(s)\text{ , large-}s})^{-\alpha(s)-\alpha(t)-1}}$$

$$\approx \Gamma(-\alpha(t))\,e^{-\alpha(t)}\,(-\alpha(s))^{\alpha(t)}$$

$$\approx \Gamma(-\alpha(t))\,e^{-\alpha(t)}\,(-\alpha')^{\alpha(t)}\,s^{\alpha(t)} \sim s^{\alpha(t)}$$

Note: for large $x$ $(x+1)! \gg x!$,

thus can't use

$(x+1)! \approx x!$

() $s \to \infty$, $t \to -\infty$, $\frac{t}{s}$ fixed.

write $t = ks$ and do similar calculation as in part b:

$$A = \frac{(-\alpha_0 - \alpha' s - 1)!\,(-\alpha_0 - k\alpha' s - 1)!}{(-2\alpha_0 - \alpha'(1+k)s - 1)!}$$

$$\approx \frac{(-\alpha_0 - \alpha' s - 1)^{-\alpha_0 - \alpha' s - 1}(-\alpha_0 - k\alpha' s - 1)^{-\alpha_0 - k\alpha' s - 1}}{(-2\alpha_0 - \alpha'(1+k)s - 1)^{-2\alpha_0 - \alpha'(1+k)s - 1}}$$

$$\times \underbrace{e^{\alpha_0 + \alpha' s + 1 + \alpha_0 + k\alpha' s + 1 - 2\alpha_0 - \alpha'(1+k)s - 1}}_{e^1}$$

$$\approx e \frac{(-\alpha' s)^{-\alpha_0 - \alpha' s - 1}(-k\alpha' s)^{-\alpha_0 - k\alpha' s - 1}}{(-\alpha'(1+k)s)^{-2\alpha_0 - \alpha'(1+k)s - 1}}$$

$$= e\,(-\alpha' s)^{-1} \frac{k^{-\alpha_0 - k\alpha' s - 1}}{(1+k)^{-2\alpha_0 - \alpha'(1+k)s - 1}}$$

does not decay as $s^{\text{number}}$

more like $\sim e^{\alpha' s}$, so does not describe pointlike particles but string!